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RESEARCH ARTICLE

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Rectifiable paths with polynomial log-signature are straight lines

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Abstract
The signature of a rectifiable path is a tensor series in the tensor algebra whose coefficients are definite iterated integrals of the path. The signature characterizes the path up to a generalized form of reparameterization. It is a classical result of Chen that the log-signature (the logarithm of the signature) is a Lie series. A Lie series is polynomial if it has finite degree. We show that the log-signature is polynomial if and only if the path is a straight line up to reparameterization. Consequently, the log-signature of a rectifiable path either has degree one or infinite support. Though our result pertains to rectifiable paths, the proof uses rough path theory, in particular that the signature characterizes a rough path up to reparameterization.

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1 | INTRODUCTION AND MAIN RESULT

Classically, a path in $\mathbb{R}^d$ is a function $\gamma : [t_0, t_1] \to \mathbb{R}^d$, with $\gamma(t)$ the value of the path at time t. We call a path *rectifiable* if it is continuous and of bounded variation. The tensor algebra $T((\mathbb{R}^d))$ is the space of tensor series over $\mathbb{R}^d$. It is isomorphic to the space of power series in d noncommuting indeterminates. The *signature* of a path is a tensor series in the tensor algebra. The entries of the signature are features that encode the path. The signature is the terminal solution (at time $t = t_1$)

to the controlled differential equation

$$d\mathbf{S} = \mathbf{S} \otimes d\gamma, \qquad \mathbf{S}(t_0) = 1 \in T\big(\big(\mathbb{R}^d\big)\big). \tag{1}$$

We denote the signature of the path γ by $\mathrm{Sig}(\gamma) := \mathbf{S}(t_1)$. It is also called the *signature transform* of γ (on $[t_0, t_1]$). The level N truncation of the tensor algebra is $T^{(N)}(\mathbb{R}^d) := \bigoplus_{k=0}^{N} (\mathbb{R}^d)^{\otimes k}$. Projecting $\mathrm{Sig}(\gamma)$ to level k gives a tensor in $(\mathbb{R}^d)^{\otimes k}$. In coordinates, its coefficient at position $(i_1, \dots, i_k)$ is the iterated integral

$$\int_{s_1=t_0}^{t_1} \left(\int_{s_2=t_0}^{s_1} \cdots \left(\int_{s_k=t_0}^{s_{k-1}} d\gamma_{i_k}(s_k) \right) \cdots d\gamma_{i_2}(s_2) \right) d\gamma_{i_1}(s_1), \tag{2}$$

with all integrals understood in classical Riemann–Stieltjes sense. The signature can be defined on any sub-interval of $[t_0, t_1]$. The *indefinite signature* of γ is the path $\mathbf{S} : [t_0, t_1] \to T((\mathbb{R}^d))$ that satisfies $\mathbf{S}(t) = \mathrm{Sig}(\gamma|_{[t_0,t]})$, where $\gamma|_{[t_0,t]}$ restricts the path γ to the interval $[t_0, t]$. The signature can be truncated to give an approximate encoding of a path. For example, truncating to level one approximates γ by the chord $\gamma(t_1) - \gamma(t_0) \in \mathbb{R}^d$.

The tensor algebra $T((\mathbb{R}^d))$ has a Lie bracket $[v, w] = v \otimes w - w \otimes v$. We denote by $[\mathbb{R}^d, \mathbb{R}^d]$ those elements of $(\mathbb{R}^d)^{\otimes 2}$ that are linear combinations of terms of the form $[v, w]$, for $v, w \in \mathbb{R}^d$. These are the skew-symmetric matrices. Extending to all levels $k \geqslant 1$, a *Lie series* is a tensor series obtained by iteratively taking Lie brackets and linear combinations. The space of Lie series is

$$L((\mathbb{R}^d)) = \mathbb{R}^d \oplus [\mathbb{R}^d, \mathbb{R}^d] \oplus [\mathbb{R}^d, [\mathbb{R}^d, \mathbb{R}^d]] + \cdots. \tag{3}$$

The logarithm of the signature is the *log-signature*, denoted $\mathrm{logSig}(\gamma)$. It is also called the *log-signature transform* of γ (on $[t_0, t_1]$). It can be obtained by evaluating the power series $\log(1 + S) = \sum_{k\geqslant 1} \frac{1}{k}(-1)^{k-1} S^{\otimes k}$ on the signature $1 + S$. It is well-known that the log-signature lies in the space of Lie series; see [6].

Example 1.1. The log-signature at level two lies in $[\mathbb{R}^d, \mathbb{R}^d]$, as follows. We compute $\mathrm{logSig}(\gamma)_{ij} = \mathrm{Sig}(\gamma)_{ij} - \frac{1}{2}\,\mathrm{Sig}(\gamma)_i\,\mathrm{Sig}(\gamma)_j$. The product rule implies $\mathrm{Sig}(\gamma)_{ij} + \mathrm{Sig}(\gamma)_{ji} = \mathrm{Sig}(\gamma)_i\,\mathrm{Sig}(\gamma)_j$. Hence, $\mathrm{logSig}(\gamma)_{ij} = -\,\mathrm{logSig}(\gamma)_{ji}$.

Like the signature, the log-signature gives an encoding of a path, which can be truncated to an approximate encoding. Log-signatures are encodings that can offer more sparsity than signatures. Moreover, they have the useful property that truncation of the log-signature preserves the property of being a Lie series. A *Lie polynomial* is a finite Lie series. We denote the space of Lie polynomials by $L(\mathbb{R}^d) \subset L((\mathbb{R}^d))$. We are interested in the paths whose log-signature is finite; that is, whose log-signature is a Lie polynomial.

Example 1.2 (Straight lines). A straight line in $\mathbb{R}^d$ is a path $\gamma : [t_0, t_1] \to \mathbb{R}^d$ with $\gamma(t) = ta$ for some $a \in \mathbb{R}^d$. Let $\gamma_a : [0, 1] \to \mathbb{R}^d$ be the straight line $\gamma_a(t) = ta$ on [0,1]. Its signature is $\mathrm{Sig}(\gamma_a) = \exp(a) := \sum_{k=0}^{\infty} \frac{1}{k!} a^{\otimes k} \in T((\mathbb{R}^d))$. That is, $\mathrm{logSig}(\gamma_a) = a \in \mathbb{R}^d \subset L((\mathbb{R}^d))$, a Lie polynomial of degree one. The straight line $\gamma(t) = ta$ on domain $[t_0, t_1]$ has signature $\mathrm{Sig}(\gamma_{(t_1-t_0)a})$, hence signatures of general straight lines are signatures of paths of the form γ_a. Signatures are unchanged under reparameterization and translation. Hence, given a vector $b \in \mathbb{R}^d$ and increasing bijection $\tau : [0, 1] \to [0, 1]$, the path $t \mapsto \gamma_a(\tau(t)) + b$ has the same signature as γ_a.

The space of paths has a concatenation product, as follows. Given a path $\gamma : [t_0, t_1] \to \mathbb{R}^d$ and a fixed $u \in [t_0, t_1]$, the path γ divides into segments $\phi := \gamma|_{[t_0,u]}$ and $\psi := \gamma|_{[u,t_1]}$. The signature of γ is the tensor product in the tensor algebra of the signatures of the two segments: $\mathrm{Sig}(\gamma) = \mathrm{Sig}(\phi) \otimes \mathrm{Sig}(\psi)$. The path γ is the concatenation of its two segments; we write this as $\gamma = \phi \star \psi$. The concatenation product $\phi \star \psi$ extends to general paths $\phi : [t_0, t_1] \to \mathbb{R}^d$ and $\psi : [s_0, s_1] \to \mathbb{R}^d$, by translating ψ and its time interval so that $s_0 = t_1$ and $\psi(s_0) = \phi(t_1)$. These transformations do not alter the signature. In particular, the concatenation of line segments $\gamma_a \star \gamma_b$ exists and has signature $\mathrm{Sig}(\gamma_a \star \gamma_b) = \exp(a) \otimes \exp(b)$.

We now describe an expression for $\mathrm{logSig}(\gamma_a \star \gamma_b)$ in terms of a and b. The *Baker–Campbell–Hausdorff (BCH) formula* [8] gives an expression for $c := \mathrm{BCH}(a, b) \in L((\mathbb{R}^d))$ such that $\exp(c) = \exp(a) \otimes \exp(b)$. That is, it gives an expression for the Lie series $\mathrm{logSig}(\gamma_a \star \gamma_b)$. The BCH formula begins

$$\mathrm{BCH}(a, b) = a + b + \frac{1}{2}[a, b] + \frac{1}{12}([a, [a, b]] + [b, [b, a]]) + \cdots . \tag{4}$$

A formulation in our context, together with explicit expression of the full commutator series can be found, for example, in [13, Section 7.3]. Note that $[a, b]$ and all higher commutators terms vanish if a and b are collinear.

We call a rectifiable path $\gamma : [t_0, t_1] \to \mathbb{R}^d$ *reduced* if there is no interval $[s_0, s_1] \subset [t_0, t_1]$ such that

$$\mathrm{Sig}(\gamma|_{[s_0,s_1]}) = 1. \tag{5}$$

Equivalently, a path is reduced if and only if the function $t \mapsto \mathbf{S}(t) = \mathrm{Sig}(\gamma|_{[t_0,t]})$ is injective, where the equivalence holds as a consequence of the multiplicative property of the signature, $\mathrm{Sig}(\gamma|_{[s,t]}) = \mathbf{S}(s)^{-1} \otimes \mathbf{S}(t)$.

Remark 1.3.

(i) Condition (5) implies that $\gamma|_{[s_0,s_1]}$ is tree-like and thus contractible to the constant path within its own image, see [3] and references therein.
(ii) Any rectifiable path has a unique reduced path with the same signature, up to reparamatrization and translation, see [15] and [3, Remark 4.1]. This reduced path minimizes the length (i.e., the 1-variation) among all rectifiable paths with the same signature.
(iii) A sufficient condition for a piecewise linear path to be reduced is that no two consecutive pieces are collinear.

We now state our main result.

Theorem 1.4. *A rectifiable reduced path has polynomial log-signature if and only if it is a straight line (or a reparameterization and translation thereof).*

This result makes progress toward understanding which Lie series can occur as the log-signatures of rectifiable paths. It relates to [18], which shows that the radius of convergence of the log-signature is finite for a large class of rectifiable paths, and conjectures a finite radius for all rectifiable paths that are not straight lines. Polynomial log-signatures would give rise to an infinite radius of convergence, so Theorem 1.4 supports this conjecture.

This paper is organized as follows. We give relevant background on rough paths in Section 2. We prove the main result in Section 3. We point out extensions to finite p-variation $p < 2$ and failure for $p \geqslant 2$ in Section 4. We relate our results to work of Boedihardjo et al. [4] in Section 5 and discuss similarities with a classical result of Marcinkiewicz in Section 6. We conclude with a polynomial perspective for piecewise linear paths in Section 7.

2 | ELEMENTS OF ROUGH PATHS

Let (E, d) be a metric space. The p-variation of a function $x : [t_0, t_1] \to (E, d)$ is

$$\sup_{\mathcal{P}} \left(\sum_{s_i \in \mathcal{P}} d(x(s_i), x(s_{i+1}))^p \right)^{\frac{1}{p}}, \tag{6}$$

where the supremum is taken over all partitions $\mathcal{P} = \{t_0 = s_0 < s_1 < \cdots < s_n = t_1\}$ of $[t_0, t_1]$.

The space of Lie series in the tensor algebra $T((\mathbb{R}^d))$ is denoted $L((\mathbb{R}^d))$, see (3). Its exponentiation $G^{(*)} = \exp(L(\left(\mathbb{R}^d\right)))$ is called the space of *group-like elements*, see [16, Theorem 2.23]. The group-like elements form a group: their products and inverses can be written as power series in the tensor algebra [16, section 2.2.1]. The projection of $G^{(*)}$ to $T^{(N)}(\mathbb{R}^d)$ is denoted $G^{(N)}$; it is the *free nilpotent Lie group of step N*, see [16, section 2.2.5], and is equipped with the metric

$$d(a, b) = \max_{k \in \{1, \ldots, N\}} \|\pi_k\left(a^{-1} \otimes b\right)\|^{\frac{1}{k}}, \tag{7}$$

where π_k is the projection of the tensor algebra $T((\mathbb{R}^d))$ or its truncation $T^{(N)}(\mathbb{R}^d)$ onto the kth graded component $(\mathbb{R}^d)^{\otimes k}$. We use this metric in the following.

Definition 2.1 (See [16, Definition 3.14]). A *weakly geometric p-rough path* is a continuous function from an interval $[t_0, t_1]$ to $G^{(\lfloor p \rfloor)}$ with finite p-variation.

The elements of $G^{(*)}$ with $\max_{k \in \mathbb{N}} \|\pi_k(a)\|^{\frac{1}{k}}$ finite are denoted $G^{(*)}_{p.r.c}$, where p.r.c. stands for positive radius of convergence [3]. We define the metric

$$d(a, b) = \max_{k \in \mathbb{N}} \|\pi_k\left(a^{-1} \otimes b\right)\|^{\frac{1}{k}}.$$

We have the following fundamental lifting theorem, see [20, Theorem 2.2.1] or [13, chapter 9]. The following formulation is from [3, Proposition 2.1].

Proposition 2.2. *Let $x : [t_0, t_1] \to G^{(\lfloor p \rfloor)}$ be a weakly geometric p-rough path. Then there is a unique continuous path $\mathbf{S} : [t_0, t_1] \to G^{(*)}$, known as the* Lyons lift, *with finite p-variation, start-point $\mathbf{S}(t_0) = 1$, and for which the projection of $\mathbf{S}|_{[t_0,t]}$ to $T^{(\lfloor p \rfloor)}(\mathbb{R}^d)$ is $x(t_0)^{-1} \otimes x(t)$. Moreover, $\mathbf{S}(t) \in G^{(*)}_{p.r.c}$.*

Weakly geometric 1-rough paths are precisely rectifiable paths. Here, the Lyons lift is the indefinite signature of the path. The following is [3, Lemma 4.6].

Lemma 2.3 (Existence and uniqueness of reduced path). *Let* $\mathbf{S} : [t_0, t_1] \to G^{(*)}_{p.r.c}$ *be a continuous path with finite p-variation. Then there exists an injective path*

$$\tilde{\mathbf{S}} : [t_0, t_1] \to G^{(*)}_{p.r.c.},$$

unique up to reparameterization, such that $\tilde{\mathbf{S}}(t_1) = \mathbf{S}(t_1)$ *and* $\tilde{\mathbf{S}}(t_0) = \mathbf{S}(t_0)$.

3 | PROOF OF MAIN RESULT

In this section, we prove Theorem 1.4. Though the statement pertains to rectifiable paths, our proof uses results from rough path theory, in particular that the signature characterizes a rough path up to a generalized form of reparameterization. We use this uniqueness for paths that are not rectifiable. We use the following lemma.

Lemma 3.1. *Let* $\ell = \ell_1 + \cdots + \ell_N$ *be a Lie polynomial of degree N, with* ℓ_k *the kth graded component. Consider the path* $[0, 1] \to T((\mathbb{R}^d))$, $t \mapsto \exp(t\ell)$. *Then*

(i) *the projection of* $\exp(t\ell)$ *to* $T^{(N)}(\mathbb{R}^d)$ *is a weakly geometric* N*-rough path;*
(ii) *the Lyons lift of this projected path is* $\exp(t\ell)$*;*
(iii) *if* $N > 1$, $\exp(t\ell)$ *cannot be the Lyons lift of a rectifiable paths in* $\mathbb{R}^d$.

Proof.

(i) The projected path takes values in $G^{(N)}$, as its logarithm is the projection of a Lie series. We use the metric (7) to show that the N-variation (6) of the path is finite. We have $x(s_i)^{-1} \otimes x(s_{i+1}) = \exp((s_{i+1} - s_i)\ell)$, where no BCH terms appear due to the collinearity of $s_i\ell$ and $s_{i+1}\ell$. We check that, for all $k \leqslant N$,

$$\sup_{\mathcal{P}} \sum_{s_i \in \mathcal{P}} \|\pi_k(\exp((s_{i+1} - s_i)\ell)\|^{\frac{N}{k}} < \infty, \tag{8}$$

as follows. The projection $\pi_k(\exp((s_{i+1} - s_i)\ell))$ is a finite linear combination

$$(s_{i+1} - s_i)^j \ell_{h_1} \otimes \cdots \otimes \ell_{h_j}, \tag{9}$$

where $h_1 + \cdots + h_j = k$. Hence, $\|\pi_k(\exp((s_{i+1} - s_i)\ell)\| \lesssim |s_{i+1} - s_i|$. Hence, (8) is finite, as $k \leqslant N$.

(ii) The path $t \mapsto \exp(t\ell)$ starts at 1 and has the right projection to $T^{(N)}(\mathbb{R}^d)$. To apply Proposition 2.2, it remains to show that it has finite N-variation; that is, that (8) holds for $k > N$. The expansion in (9) still holds, where each factor ℓ_h has degree at most N. Hence, $\|\pi_k(\exp((s_{i+1} - s_i)\ell)\| \lesssim |s_{i+1} - s_i|^{\frac{k}{N}}$ and (8) is finite.†

(iii) If the path $\exp(t\ell)$ is the Lyons lift of a rectifiable path in $\mathbb{R}^d$, projecting to $\mathbb{R}^d$ shows that it must be the lift of the path $\gamma(t) = t\ell_1$. The lift is then $\exp(t\ell_1)$, which differs from $\exp(t\ell)$ whenever $N > 1$. □

We now prove Theorem 1.4.

† Such computations can by bypassed by viewing geometric rough paths as Cartan developments of (here, linear) paths in the Lie algebra, see [2, Proposition 2.14].

Proof of Theorem 1.4. Without loss of generality, our paths are defined on [0,1]. The straight line $\gamma_a : [0,1] \to \mathbb{R}^d$ with $\gamma_a(t) = at$ has log-signature the finite Lie polynomial $a \in \mathbb{R}^d$, see Example 1.2. The log-signature of a path γ that is a reparameterization or translation of γ_a is also the finite Lie polynomial $a \in \mathbb{R}^d$. This proves one direction.

For the converse, take a reduced rectifiable path $\gamma : [0,1] \to \mathbb{R}^d$ and assume that $\ell := \mathrm{logSig}(\gamma)$ is a Lie polynomial with $N = \deg \ell < \infty$. We show that $N = 1$. Let $\mathbf{S}$ be the Lyons lift of γ. The path $\mathbf{S}$ is injective, as γ is reduced. By assumption, we have $\mathbf{S}(0) = 1$, and $\mathbf{S}(1) = \exp(\ell)$.

Set $\mathbf{X}(t) = \exp(t\ell)$. It takes values in $G^{(*)}_{p.r.c}$ and defines, under projection to $T^{(N)}(\mathbb{R}^d)$, a weakly geometric N-rough path, by Lemma 3.1(i). Like $\mathbf{S}$, the path $\mathbf{X}$ sends $[0,1]$ injectively into $G^{(*)}_{p.r.c}$ and satisfies $\mathbf{X}(0) = 1$, $\mathbf{X}(1) = \exp(\ell)$. Hence, $\mathbf{X}$ is a reparameterization of $\mathbf{S}$, by Lemma 2.3. That is, the equality

$$\mathbf{X}(t) = \mathbf{S}(\tau(t)) = \mathrm{Sig}((\gamma \circ \tau)|_{[0,t]}) =: \mathrm{Sig}(\tilde{\gamma}|_{[0,t]}),$$

holds in $T((\mathbb{R}^d))$ for all $t \in [0,1]$, where $\tilde{\gamma}$ is a rectifiable path on $\mathbb{R}^d$ obtained by reparameterization of γ under τ. Hence, $\mathbf{X}$ is the Lyons lift of $\tilde{\gamma}$. Hence, $N = 1$, by Lemma 3.1(iii). □

Remark 3.2. The proof of Theorem 1.4 breaks if $\ell := \mathrm{logSig}(\gamma)$ is infinite: in this case the path $\mathbf{X}(t) = \exp(t\ell) \in T((\mathbb{R}^d))$ fails to have finite p-variation for some p. Hence, Lemma 2.3 does not apply to $\mathbf{X}$, and we cannot deduce that $\mathbf{S}$ and $\mathbf{X}$ agree up to reparameterization.

Theorem 1.4 says that any rectifiable reduced path that is not a straight-line (or reparameterization and translation thereof) has infinitely many nonzero terms in its log-signature. Consequently, the log-signature of a rectifiable path either has degree one or infinite support. We give a second proof via *smooth rough paths* [2], which bypasses some analytic details of the first proof. Smooth rough paths are in precise analogy with *smooth models* in Hairer's regularity structures [5, Definition 6.7].

Alternative proof of Theorem 1.4. We assume for simplicity that γ is smooth, but the argument is identical for continuously differentiable γ and, more generally, absolutely continuous γ, writing t-almost surely when doing calculus. Starting with a general rectifiable path γ, reparameterization by running length does not alter the signature and yields an absolutely continuous paths, so there is no loss of generality.

Let N be the degree of Lie polynomial ℓ. Let $\mathbf{Z}(t)$ denote the projection of $\exp(t\ell)$ to $T^{(N)}(\mathbb{R}^d)$. It is a level-N smooth geometric rough path in the sense of [2]. By the fundamental theorem of smooth geometric rough paths [2, Theorem 2.8], it has a unique extension $\mathbf{X}(t) \in T((\mathbb{R}^d))$ that satisfies the minimality condition that $\mathbf{X}^{-1}(t) \otimes \dot{\mathbf{X}}(t)$ lies in the truncated space $T^{(N)}(\mathbb{R}^d)$ rather than $T((\mathbb{R}^d))$. The extension therefore satisfies $\mathbf{X}^{-1}(t) \otimes \dot{\mathbf{X}}(t) = \mathbf{Z}^{-1}(t) \otimes \dot{\mathbf{Z}}(t) \in T^{(N)}(\mathbb{R}^d)$. In our case the right-hand side is $\ell \in T^{(N)}(\mathbb{R}^d)$ and solving the ODE in $T((\mathbb{R}^d))$ with $\mathbf{X}(0) = 1$, gives $\mathbf{X}(t) = \exp(t\ell)) \in T^{(N)}(\mathbb{R}^d)$. Consistency of $\mathbf{Z}$ with the Lyons' extension is verified in [2, Proposition 2.14].

We view γ as a level-1 smooth geometric rough path, whose unique minimal extension is $\mathbf{S}(t)$, the indefinite signature of γ. Again we rely on Lemma 2.3. (Specializing it to the setting of smooth geometric rough paths does not seem to lead to major simplifications.) As in our previous proof, it follows that $\mathbf{X}$ and $\mathbf{S}$ are reparameterizations of each other and, as Riemann–Stieltjes integration is not affected by such reparameterizations, we have, using $d\gamma = \mathbf{S}^{-1} \otimes d\mathbf{S}$,

$$\ell = \int_0^1 \mathbf{X}(t)^{-1} \otimes d\mathbf{X}(t) = \int_0^1 \mathbf{S}(t)^{-1} \otimes d\mathbf{S}(t) = \int_0^1 d\gamma = \gamma(1) - \gamma(0) \in \mathbb{R}^d.$$

In particular, $\ell \in \mathbb{R}^d$. □

4 | BEYOND RECTIFIABILITY

Theorem 1.4 generalizes from rectifiable paths to continuous paths of finite p-variation, provided $p < 2$. The signature transform remains well-defined by iterated Young integration, see, for example, [13, 17]. We have the following generalization of Theorem 1.4 with the same proof.

Theorem 4.1. *Let $p < 2$. A continuous reduced path of finite p-variation has polynomial log-signature if and only if it is a straight line (or a reparameterization and translation thereof).*

Theorem 4.1 is false if $p \geqslant 2$. Indeed, it suffices to consider $t \mapsto \exp(t\ell) \in T^{(N)}(\mathbb{R}^d)$ for ℓ a nonzero homogenous Lie polynomial of degree $N \geqslant 2$. Such rough paths are called *pure-N rough paths*, or pure area rough paths when $N = 2$.

5 | PROOF VIA BGS ESTIMATES

Another route to Theorem 1.4 was suggested to us by Boedihardjo, in terms of a careful application of the analytic estimates of [4, Theorem 2.2]. This replaces the use of [3, Lemma 4.6] that was central to our two earlier proofs. All three proofs use results from rough path theory. It is an open problem to prove Theorem 1.4 via an approach that studies only rectifiable paths.

Yet another proof of Theorem 1.4. Given a tensor series $\mathbf{T} \in T((\mathbb{R}^d))$, we define

$$L_p(\mathbf{T}) := \limsup_{k\to\infty} \left(\left(\frac{k}{p}\right)! \|\pi_k(\mathbf{T})\|\right)^{p/k}, \tag{10}$$

where $\|\cdot\|$ denotes a norm of order k tensors, for any k, that is compatible in the sense that $\|a \otimes b\| \leqslant \|a\|\,\|b\|$. Applied to the signature of γ, we see that

$$L_1(\mathrm{Sig}(\gamma)) \leqslant \|\gamma\|_{1\text{-var}} \quad \text{and} \quad L_p(\mathrm{Sig}(\gamma)) = 0 \text{ for } p > 1, \tag{11}$$

as follows. This is a consequence of

$$\|\pi_k(\mathrm{Sig}(\gamma))\| = \frac{1}{k!}\left\|\int_{[0,1]^k} \dot\gamma \otimes \cdots \otimes \dot\gamma\, d(t_1,\dots,t_k)\right\| \leqslant \frac{1}{k!}\int_{[0,1]^k} \|\dot\gamma \otimes \cdots \otimes \dot\gamma\| d(t_1,\dots,t_k)$$

$$\leqslant \frac{1}{k!}\int_{[0,1]^k} \|\dot\gamma\| \cdots \|\dot\gamma\| d(t_1,\dots,t_k) = \frac{1}{k!}\left(\int_0^1 \|\dot\gamma\|\,\mathrm{dt}\right)^k = \frac{1}{k!}\|X\|^k_{1-\mathrm{var}}.$$

Here $d\gamma = \dot{\gamma}dt$, which entails no loss of generality. Reparameterization by running length does not alter the signature and yields an absolutely continuous path. Alternatively, use facts on Stieltjes integrals that directly justify the above estimation.

The result [4, Theorem 2.2] states that (under the additional assumption that the tensor norms are projective, which entails no loss of generality in our finite-dimensional setup, with base space $\mathbb{R}^d$) for each $m \geqslant 1$, there exists a constant $c(m,d) \in (0,1]$ such that

$$c(m,d)\|\pi_m(\ell)\| \leqslant L_m(\mathrm{Sig}(\mathbf{X})) \leqslant \|\pi_m(\ell)\|$$

for every pure-m rough path $\mathbf{X}(t) = \exp(t\ell) \in T^{(m)}(\mathbb{R}^d)$, with $\deg \ell = m$. We know $\mathrm{Sig}(\mathbf{X})$ is $\exp(\ell) \in T((\mathbb{R}^d))$, by [4, Proposition 2.2]. This implies the estimate

$$c(m,d)\|\pi_m(\ell)\| \leqslant L_m(\exp(\ell)) \leqslant \|\pi_m(\ell)\|. \tag{12}$$

If $\mathrm{Sig}(\gamma) = \exp(\ell)$, for some Lie polynomial ℓ of degree m, then (11) and (12) implies $m = 1$. It follows that $\ell \in \mathbb{R}^d$ and we can conclude as before. □

6 | CONNECTIONS TO MARCINKIEWICZ'S THEOREM

Theorem 1.4 has the following implication.

Corollary 6.1. *Let P be a Lie polynomial of degree $m \geqslant 2$. Then P cannot be the log-signature transform of a rectifiable path.*

We compare this to a classical result of Marcinkiewicz [21], see also [19, Theorem B], which we restate in a form that exhibits their similarity. Let $X = X(\omega)$ be a random variable with values in $\mathbb{R}^d$, with finite moments of all orders. Its *moment transform* is

$$\mu(X) := \mathbb{E}(\exp(X)) = 1 + \sum_{k\geqslant 1} \frac{1}{k!}\mathbb{E}\left(X^{\otimes k}\right).$$

The moment transform lies the symmetric algebra over $\mathbb{R}^d$, denoted $\mathrm{Sym}((\mathbb{R}^d))$, which consists of series of symmetric tensors. This space is isomorphic to the space of power series in d commuting indeterminates. The *log-moment* (or *cumulant*) transform is

$$\kappa(X) := \log \mu(X) \in \mathrm{Sym}\left(\left(\mathbb{R}^d\right)\right).$$

The entries of $\mu(X)$ and $\kappa(X)$ are the multivariate moments and cumulants of X, up to factorial constants.

Example 6.2. If $X \sim N(b,a)$ is normally distributed with mean $b \in \mathbb{R}^d$ and covariance $a \in \mathbb{R}^d \otimes \mathbb{R}^d$, the log-moment transform is $\kappa(X) = b + a/2$, a degree two polynomial in $\mathrm{Sym}((\mathbb{R}^d))$.

Theorem 6.3 [21]. *Let P be a polynomial of degree $m \geqslant 3$. Then P cannot be the log-moment transform of a probability measure (with all moments finite).*

We discuss steps toward a mutual generalization of Corollary 6.1 and Theorem 6.3. Let $p \in [1,\infty)$. Given a random weakly geometric p-rough path $\mathbf{X} = \mathbf{X}(\omega)$ in $\mathbb{R}^d$, its signature and log-signature are $T((\mathbb{R}^d))$-valued random variables. Assuming (componentwise) integrability, we define the *expected signature* and *signature cumulant* by

$$\boldsymbol{\mu}(\mathbf{X}) := \mathbb{E}(\mathrm{Sig}(\mathbf{X})), \quad \boldsymbol{\kappa}(\mathbf{X}) := \log \mathbb{E}(\mathrm{Sig}(\mathbf{X})) \in T\left(\left(\mathbb{R}^d\right)\right).$$

The signature cumulant of the Brownian rough path is finite, as follows.

Example 6.4. Let $X \sim \mathrm{Bm}(a, b)$, meaning that $X = (X(t) : 0 \leqslant t \leqslant 1)$, is a Brownian motion with drift b and covariance a. That is, $Y(t) := X(t) - bt$ defines a centered Gaussian process with covariance $\mathbb{E}(Y(s) \otimes Y(t)) = a \min\{s, t\}$; in particular $X(t) \sim N(bt, at)$. Let $\mathbf{X} \sim \mathrm{Brp}(b, a)$ by which we mean that $\mathbf{X}$ is the *Brownian rough path* obtained from X by iterated Stratonovich integration. Then $\boldsymbol{\mu}(\mathbf{X}) = \exp(b + a/2)$. See, for example, [10, chapter 3], [1, section 7.1] for an algebraic geometry perspective, and [11] for a far-reaching extension to a general semimartingale. Equivalently,

$$\boldsymbol{\kappa}(\mathbf{X}) := b + a/2,$$

which is a degree two polynomial in $T((\mathbb{R}^d))$.

The naive guess is that the signature cumulant of any Gaussian rough path (e.g., in the sense of [13, chapter 15]) is finite. This is wrong, by the following example.

Example 6.5. Take independent $\mathbf{X}^i \sim \mathrm{Brp}(0, a_i)$, for $i = 1, 2$. Write $\mathbf{X}^1 \star \mathbf{X}^2$ for the concatenation of these random rough paths. The expected signature is $\exp(a_1/2) \otimes \exp(a_2/2)$. By the BCH formula,

$$\boldsymbol{\kappa}(\mathbf{X}^1 \star \mathbf{X}^2) = \mathrm{BCH}(a_1/2, a_2/2) = a_1/2 + a_2/2 + [a_1, a_2]/8 + \cdots \in T\left(\left(\mathbb{R}^d\right)\right).$$

Unless a_1 and a_2 are collinear, this is nonpolynomial, see Corollary 7.2.

The above example does not have stationarity of increments, in contrast to Example 6.4. See, for example, [13, chapter 13] or [12].

Definition 6.6 (Brownian-like signatures). Given a random weakly geometric p-rough path $\mathbf{X} = \mathbf{X}(\omega)$ we call its signature $\mathrm{Sig}(\mathbf{X})$ *Brownian like* if its signature cumulants $\boldsymbol{\kappa}(\mathbf{X})$ are well-defined and polynomial, that is, finite in $T((\mathbb{R}^d))$.

For example, [9] computed the expected signature of the so-called magnetic Brownian (rough) path to be of the form $\exp(\tilde{b} + a/2)$ where $\tilde{b} = b^1 + b^2 \in L^2(\mathbb{R}^d)$ and $a \in (\mathbb{R}^d)^{\otimes 2}$ is a symmetric matrix. Note that $\tilde{b} + a/2$ is a polynomial of degree two in $T((\mathbb{R}^d))$. We also note the example of the deterministic weakly geometric N-rough path, given by $t \mapsto \exp(t\ell) \in T^{(N)}(\mathbb{R}^d)$, for $\ell \in L(\mathbb{R}^d)$ of degree N, which has $\boldsymbol{\kappa} = \mathrm{logSig}$ equal to the Lie polynomial ℓ. We suspect that Brownian like signature are related to signatures of "higher order" Brownian rough paths. This is left to future work.

7 PIECEWISE LINEAR PATHS

We specialize to piecewise linear paths, studying connections to systems of polynomial equations. The piecewise linear path with m pieces $v_1, \dots, v_m \in \mathbb{R}^d$ is the concatenation $\gamma := \gamma_{v_1} \star \gamma_{v_2} \star \cdots \star \gamma_{v_m}$.

Corollary 7.1. *Let γ be a piecewise linear path with m pieces and no two consecutive pieces collinear. The log-signature of γ is finite if and only if $m = 1$.*

Proof. A sufficient for a piecewise linear path to be reduced is that no two consecutive pieces are collinear. Hence, γ is a straight line, by Theorem 1.4. As consecutive pieces are not collinear, the straight line has $m = 1$ piece. □

We rephrase Corollary 7.1 as a statement about the Baker Campbell Hausdorff formula. Given $v_1, \dots, v_m \in \mathbb{R}^d$, we define the iterated BCH formula to be the Lie series $c := \mathrm{BCH}(v_1, \dots, v_m) \in L((\mathbb{R}^d))$ such that $\exp(c) = \exp(v_1) \otimes \cdots \otimes \exp(v_m)$. When $m = 1$, we set $\mathrm{BCH}(v_1) = v_1$. For $m = 2$, see (4). We suspect that the following may be known to experts in the Lie algebra community.

Corollary 7.2. *Fix $v_1, \dots, v_m \in \mathbb{R}^d$ with no consecutive v_i collinear. The iterated Baker Campbell Haussdorff formula $\mathrm{BCH}(v_1, \dots, v_m) \in L((\mathbb{R}^d))$ has finitely many nonzero terms if and only if $m = 1$.*

Proof. The log-signature of the piecewise linear path with pieces $v_1, \dots, v_m$ is $\mathrm{BCH}(v_1, \dots, v_m)$, which is finite if and only if $m = 1$, by Corollary 7.1. □

The task of recovering a path from its truncated signature was studied via solving systems of polynomial equations in [1, 22]. We apply this perspective to Corollaries 7.1 and 7.2. Consider the piecewise linear path with pieces $v_1, \dots, v_m \in \mathbb{R}^d$. Its log-signature at level k is a tensor in $(\mathbb{R}^d)^{\otimes k}$ that is a rational linear combination of terms $v_{i_1} \otimes \cdots \otimes v_{i_k}$ for some $i_1, \dots, i_k \in \{1, \dots, m\}$.

Example 7.3. The log-signature at level two is

$$\frac{1}{2} \sum_{1 \leqslant i < j \leqslant m} (v_i \otimes v_j - v_j \otimes v_i). \tag{13}$$

The log-signature at level three is

$$\frac{1}{12} \sum_{i \neq j} \left(v_i^{\otimes 2} \otimes v_j + v_i \otimes v_j^{\otimes 2} \right) + \frac{1}{3} \sum_{(i,j,k) \in S_1} v_i \otimes v_j \otimes v_k - \frac{1}{6} \sum_{(i,j,k) \in S_2} v_i \otimes v_j \otimes v_k, \tag{14}$$

where $S_1 = \{(i, j, k) : i < j < k \text{ or } i > j > k\}$, $S_2 = \{(i, j, k) : i < j > k \text{ or } i > j < k\}$.

Each entry of the level k log-signature is a degree k polynomial in the md entries v_{ij} of the m pieces $v_i \in \mathbb{R}^d$. Hence, for piecewise linear paths, Theorem 1.4 describes the real solutions to an (infinite) system of polynomial equations. In fact, the vanishing of finitely many levels of the log-signature suffice to conclude that the path is a straight line.

Corollary 7.4. *Given $n_1 \geqslant 1$ and $m \geqslant 2$, there exists a smallest integer $n_2 = n_2(n_1, m) \geqslant n_1$ such that there is no piecewise linear path with m pieces, and no consecutive pieces collinear, whose log-signature vanishes at levels $n_1, n_1 + 1, \dots, n_2$.*

Proof. Fix a piecewise linear path with pieces $v_1, \dots, v_m \in \mathbb{R}^d$. Each entry of the log-signature is a polynomial in $\mathbb{Q}[v_{ij} : 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant d]$. Let $I_{n_1,k}$ be the ideal generated by the entries of log-signature at levels $n_1, n_1 + 1, \dots, k$, for $k \geqslant n_1$. As k increases, we obtain $I_{n_1,k} \subseteq I_{n_1,k+1} \subseteq \cdots$, which stabilizes at some ideal I_{n_1,n_2}, by Noetherianity, see, for example, [7, section 5, Theorem 7]. The log-signature vanishes at level n_1 and above whenever the pieces lies in the vanishing locus of I_{n_1,n_2}. Hence, the vanishing locus of I_{n_1,n_2} contains no real v_i with no consecutive pieces collinear, by Corollary 7.1, as $m \geqslant 2$.

We rule out dependence of n_2 on d. Assume there is a path in $\mathbb{R}^d$, where $d \geqslant 2$, with m pieces, no two consecutive pieces collinear, whose log-signature at levels $n_1, n_1 + 1, \dots, n_2$ vanishes. Embedding the path into $\mathbb{R}^{d'}$ for $d' > d$ gives a path with this property in $\mathbb{R}^{d'}$. For $2 \leqslant d' < d$, there exist projections of the path onto $\mathbb{R}^{d'}$ in which consecutive increments remain noncollinear, as follows. A general linear projection $A : \mathbb{R}^d \to \mathbb{R}^{d'}$ has kernel of dimension $d - d'$ and $A(v_i) \neq 0$ for all pieces v_i. If $A(v_i)$ and $A(v_{i+1})$ are collinear, then $\ker A$ intersects the one-dimensional space $\{v_i - \lambda v_{i+1} : \lambda \in \mathbb{R}\}$. Such intersections do not occur in general, based on a dimension count, provided $(d - d') + 1 < d$; that is, provided $d' \geqslant 2$.

The log-signature of the projection also vanishes at levels $n_1, n_1 + 1, \dots, n_2$. Hence, n_2 is a function of just n_1 and m. □

Corollary 7.4 suggests computing the required upper level n_2 for different starting levels n_1 and numbers of pieces m. We leave the study of the function $n_2 = n_2(n_1, m)$ to future work and conclude this article with two examples.

Example 7.5 (Two pieces). Let $n_1 = 2$. The level two log-signature is $\frac{1}{2}[v_1, v_2]$, which vanishes if and only if v_1 and v_2 are collinear. Hence, $n_2(2, 2) = 2$: any path with two pieces whose log-signature vanishes at level two is a straight line. Now we consider $n_1 = 3$. The level three log-signature is

$$[v_1, [v_1, v_2]] + [v_2, [v_2, v_1]],$$

which vanishes if and only if v_1, v_2 are collinear. Hence, $n_2(3, 2) = 3$: no piecewise linear path with two noncollinear pieces has vanishing level three log-signature.

Example 7.6 (Three pieces). Let $n_1 = 2$. There exist paths with three pieces, no consecutive pieces collinear, whose level two signature vanishes. For example, let

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} a \\ 1 \end{bmatrix},$$

for any $a \in \mathbb{R}$. Hence, $n_2(2, 3) \geqslant 3$. Setting (13) and (14) to zero gives the ideal $I_{2,3}$ from the proof of Corollary 7.4. A Macaulay2 [14] computation shows that $I_{2,3}$ has three components: $v_1 + v_2 = 0$, $v_3 + v_2 = 0$ and v_1, v_2, v_3 all parallel. In all three components, there exist two consecutive pieces that are collinear. In the first two components, the path is not reduced and the corresponding reduced path is a straight line. In the third component, the path is a straight line. Hence, all components lie in $I_{2,k}$ for all $k \geqslant 2$, and therefore $n_2(2, 3) = 3$. Next we consider $n_1 = 3$. Setting (14) to zero reveals that the third order log-signature vanishes for paths with pieces v_1, v_2, v_3 that satisfy $v_1 + 3v_2 + v_3 = 0$. Hence, $n_2(3,3) \geqslant 4$.

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