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Identifiability of overcomplete independent component analysis

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Independent component analysis (ICA) studies mixtures of independent latent sources. An ICA model is identifiable if the mixing can be recovered uniquely. When the number of sources is at most the number of observations, Comon proved in 1994 that ICA is identifiable if and only if at most one source is Gaussian. However, in the overcomplete setting, where the number of sources exceeds the number of observations, an if and only if characterization for identifiability has been missing. In this paper, we give such a characterization. The proof studies linear spaces of rank one symmetric matrices. For generic mixing, we present an identifiability condition in terms of the number of sources and the number of observations. We use our identifiability results to design a coupled matrix and tensor decomposition algorithm to recover the mixing matrix from data and apply it to synthetic data and two real datasets.

Keywords: Independent component analysis; identifiability; real algebraic geometry

1. Introduction

Blind source separation seeks to recover latent sources and unknown mixing from observations of mixtures of signals [10]. A special case is independent component analysis (ICA), which assumes that the latent sources are independent. Classical ICA assumes that the observations are linear mixtures of the independent sources. That is,

$$\mathbf{x} = A\mathbf{s}, \tag{1}$$

where $\mathbf{s} = (s_1, \ldots, s_J)^\top$ is a vector of independent sources, $\mathbf{x} = (x_1, \ldots, x_I)^\top$ collects the observed variables, and $A \in \mathbb{R}^{I \times J}$ is an unknown mixing matrix. Equation (1) says that the distribution of $\mathbf{x}$ is the pushforward of the distribution of $\mathbf{s}$ under the linear map A. We assume access to the full distribution of $\mathbf{x}$ and defer the finite sample setting to Section 6.

Applications of ICA include recovering speech signals [4] and brain signals [23], causal discovery [39], and image decomposition [22,35]. The ICA framework has seen extensions to nonlinear mixtures, see e.g. [24].

The ICA model (1) is identifiable if the mixing matrix A can be uniquely recovered from the distribution of $\mathbf{x}$, up to column scaling and permutation. Identifiability is crucial to interpreting the entries of the mixing matrix. Depending on the application, these encode causal relationships [39] or image components [22,35]. Scaling and permutation indeterminacy are unavoidable, corresponding to the arbitrary order and scale of the sources, which does not affect their independence[1].

The following characterization of the identifiability of ICA is well known. Recall that a distribution is non-degenerate if it is not supported at a single point.

Theorem 1.1 (See [9, Theorem 11 and Corollary 13]). *Consider the ICA model* $\mathbf{x} = A\mathbf{s}$*, where* $\mathbf{s} = (s_1, \ldots, s_J)^\top$ *is a vector of non-degenerate independent sources,* $\mathbf{x} = (x_1, \ldots, x_I)^\top$ *is a vector of*

[1]Let $B = APD$, for permutation matrix P and diagonal matrix D. Then $A\mathbf{s} = B\mathbf{r}$, where $\mathbf{r} = D^{-1}P^\top\mathbf{s}$ reorders and scales $\mathbf{s}$.

observations, and $A \in \mathbb{R}^{I\times I}$ is invertible. Identifiability holds if and only if at most one of the sources is Gaussian.

Theorem 1.1 stems from the connection between linear transformations of independent variables and Gaussianity.

Theorem 1.2 (The Darmois–Skitovich theorem [11,40,41]). *Let $s_1, \dots, s_I$ be non-degenerate independent random variables. If the linear combinations $\sum_{i=1}^{I} \lambda_i s_i$ and $\sum_{j=1}^{I} \mu_j s_j$ are independent, then any s_i with $\lambda_i \mu_i \neq 0$ is Gaussian.*

Theorem 1.1 resolves the identifiability of ICA when the number of sources and observations are equal, the case of square mixing matrix $A \in \mathbb{R}^{I\times I}$. It extends to the case of fewer sources than observations, provided the mixing matrix has full rank, see [16, Theorem 3].

Our goal in this paper is to give a characterization of the identifiability of ICA that does not restrict the number of sources and observations. That is, we seek to generalize Theorem 1.1 to overcomplete ICA, where there are more sources than observations. Overcomplete ICA appears in sparse coding and finding signals in speech data [29], as well as decomposing images [22,34]. Algorithms for overcomplete ICA include [12,35,43].

To date, a characterization of the identifiability of overcomplete ICA has been missing. The following partial results are known. If no source is Gaussian, then (1) is identifiable if and only if no pair of columns in A are collinear [16, Theorem 3]. If there are at least two Gaussian sources, non-identifiability holds like in the square case, as follows. Suppose that sources s_1 and s_2 are Gaussian, with variances σ_1 and σ_2, respectively. Let $u_1 = \lambda_1 s_1 + \lambda_2 s_2$ and $u_2 = \mu_1 s_1 + \mu_2 s_2$. The variables u_1, u_2 are Gaussian, hence they are independent if and only if they are uncorrelated. Fix non-zero λ_i and μ_j such that $\mathbb{E}[u_1 u_2] - \mathbb{E}[u_1]\mathbb{E}[u_2] = \lambda_1\mu_1\sigma_1 + \lambda_2\mu_2\sigma_2 = 0$ and define $\nu = \lambda_1\mu_2 - \lambda_2\mu_1$. Let $\mathbf{r} := (\nu u_1, \nu u_2, s_3, \dots, s_J)^\top$. Then $A\mathbf{s}$ and $B\mathbf{r}$ have the same distribution, where

$$B = \begin{pmatrix} \vdots & \vdots & \vdots & & \vdots \\ \mu_2\mathbf{a}_1 - \mu_1\mathbf{a}_2 & -\lambda_2\mathbf{a}_1 + \lambda_1\mathbf{a}_2 & \mathbf{a}_3 & \cdots & \mathbf{a}_J \\ \vdots & \vdots & \vdots & & \vdots \end{pmatrix},$$

and $\mathbf{a}_1, \dots, \mathbf{a}_J$ are the columns of $A \in \mathbb{R}^{I\times J}$. Matrices A and B are not the same up to permutation and scaling, hence identifiability does not hold.

To characterize the identifiability of overcomplete ICA, it remains to settle the case where a single source is Gaussian. Since identifiability is impossible with more than two Gaussian sources, we make the following definition.

Definition 1.3. A mixing matrix $A \in \mathbb{R}^{I\times J}$ is *identifiable* if for any non-degenerate sources $\mathbf{s} = (s_1, \dots, s_J)^\top$ with at most one s_j Gaussian, the matrix A can be recovered uniquely, up to permutation and scaling of its columns, from $A\mathbf{s}$. That is, if $A\mathbf{s}$ and $B\mathbf{r}$ have the same distribution for some $B \in \mathbb{R}^{I\times K}$ with $K \leq J$ and some $\mathbf{r} = (r_1, \dots, r_K)$, with the same number of Gaussian entries as $\mathbf{s}$, then $J = K$ and matrices A and B coincide, up to permutation and scaling of columns.

Remark 1.4 (Recovering mixing vs. sources). In causal inference [39], the mixing matrix reveals the causal relationships between variables, while the source variables give the distributions of the exogenous noise. In image decomposition [22,35], the mixing matrix gives the image components, while the source

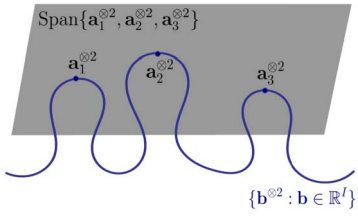

Figure 1. Illustration of Theorem 1.5.

variables follow Bernoulli distributions. One can recover the distribution of the sources $\mathbf{s}$ from the mixing A and the distribution of the observed variables $\mathbf{x}$, provided A has a left inverse. This holds for full rank matrices $A \in \mathbb{R}^{I \times J}$ with $J \leq I$. Once $J > I$, we can no longer recover the sources, but may still recover the mixing matrix.

In this paper, identifiability refers to Definition 1.3. Given vector $\mathbf{v} \in \mathbb{R}^I$, the rank one matrix $\mathbf{v}\mathbf{v}^\mathsf{T}$ is denoted $\mathbf{v}^{\otimes 2}$. Our first contribution is an if and only if characterization of the identifiability of ICA, with no restrictions on the number of sources or observations. See Figure 1 for an illustration.

Theorem 1.5. *Fix $A \in \mathbb{R}^{I \times J}$ with columns $\mathbf{a}_1, \ldots, \mathbf{a}_J$ and no pair of columns collinear. Then A is identifiable if and only if the linear span of $\mathbf{a}_1^{\otimes 2}, \ldots, \mathbf{a}_J^{\otimes 2}$ does not contain any real matrix $\mathbf{b}^{\otimes 2}$ unless $\mathbf{b}$ is collinear to $\mathbf{a}_j$ for some $j \in \{1, \ldots, J\}$.*

The d-th cumulant of a distribution on $\mathbb{R}^I$ is a symmetric order d tensor of format $I \times \cdots \times I$ that encodes properties of the distribution [31, Chapter 2]. The d-th cumulant of $\mathbf{x} = A\mathbf{s}$ is

$$\kappa_d = \sum_{j=1}^{J} \lambda_j \mathbf{a}_j^{\otimes d},$$

where $\mathbf{s} = (s_1, \ldots, s_J)$ has independent entries, the scalar λ_j is the d-th cumulant of s_j, and $\mathbf{a}_j^{\otimes d}$ is the tensor with $(i_1, \ldots, i_d)$ entry $(\mathbf{a}_j)_{i_1} \cdots (\mathbf{a}_j)_{i_d}$. This follows from the fact that the cumulant tensor of a vector of independent entries is diagonal and from the multilinearity property of cumulants.

Theorem 1.5 may be surprising at first, since it only uses second order information about the matrix A. We might have expected a condition that involves terms $\mathbf{a}_j^{\otimes d}$ from higher-order cumulants. However, since for a Gaussian all cumulants of order greater than two are zero, our characterization turns out to only depend on the second powers $\mathbf{a}_j^{\otimes 2}$.

Theorem 1.5 implies the identifiability of square ICA, as follows.

Example 1.6. Let $\mathbf{x} = A\mathbf{s}$, where $A \in \mathbb{R}^{I \times I}$ is invertible and $\mathbf{s}$ is a vector of independent sources with one source Gaussian. Assume there exists $\mathbf{b} \in \mathbb{R}^I$ with $\mathbf{b}^{\otimes 2} = \sum_{i=1}^{I} \lambda_i \mathbf{a}_i^{\otimes 2}$. After a change of basis, we have $(\mathbf{b}')^{\otimes 2} = \sum_{i=1}^{I} \lambda_i \mathbf{e}_i^{\otimes 2}$, where the $\mathbf{e}_i$ are elementary basis vectors, since the columns $\mathbf{a}_i$ are linearly independent. Hence $\mathbf{b}'$ is a diagonal rank one matrix. Therefore $\mathbf{b}'$ is parallel to $\mathbf{e}_i$ for some i, so $\mathbf{b}$ is parallel to $\mathbf{a}_i$ for some i. The model is then identifiable, by Theorem 1.5.

The following examples illustrate Theorem 1.5 in overcomplete settings.

Example 1.7. Consider the mixing matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}. \tag{2}$$

No pair of columns of A are collinear. Let $\mathbf{b}^{\otimes 2} = \sum_{j=1}^{6} \lambda_j \mathbf{a}_j^{\otimes 2}$. Then

$$\mathbf{b}^{\otimes 2} = \begin{bmatrix} \lambda_1 + \lambda_5 & \lambda_5 & \lambda_5 & 0 \\ \lambda_5 & \lambda_2 + \lambda_5 + \lambda_6 & \lambda_5 + \lambda_6 & \lambda_6 \\ \lambda_5 & \lambda_5 + \lambda_6 & \lambda_3 + \lambda_5 + \lambda_6 & \lambda_6 \\ 0 & \lambda_6 & \lambda_6 & \lambda_4 + \lambda_6 \end{bmatrix}.$$

The 2×2 minors of this matrix vanish, since $\mathbf{b}^{\otimes 2}$ has rank one. This cannot happen unless all but one λ_i is zero, as can be seen from a Macaulay2 [18] computation, so $\mathbf{b}$ is collinear to one of the $\mathbf{a}_j$. Hence A is identifiable, by Theorem 1.5.

To explain the condition in Theorem 1.5, we show directly that A is identifiable. To simplify our exposition, we assume that the non-Gaussian sources have non-vanishing fourth cumulants. Suppose $s_1, \ldots, s_5$ are the non-Gaussian sources. A tensor of the form $\sum_{j=1}^{5} \lambda_j \mathbf{a}_j^{\otimes 4}$ has a unique tensor decomposition, by Kruskal's criterion [28]. Hence columns $\mathbf{a}_1, \ldots, \mathbf{a}_5$ can be recovered uniquely, up to permutation and scaling. The covariance matrix of $A\mathbf{s}$ has the form $\sum_{j=1}^{6} \mu_j \mathbf{a}_j^{\otimes 2}$. If there are two candidates $\mathbf{a}_6$ and $\mathbf{b}$ for the last column, then $\mathbf{b}^{\otimes 2} \in \mathrm{Span}\{\mathbf{a}_j^{\otimes 2} : j = 1, \ldots, 6\}$.

Example 1.8. The mixing matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \tag{3}$$

does not satisfy the condition in Theorem 1.5, since $\mathbf{b}^{\otimes 2} = -(\mathbf{a}_1)^{\otimes 2} + 2(\mathbf{a}_2)^{\otimes 2} + 2(\mathbf{a}_3)^{\otimes 2}$ holds for $\mathbf{b} = (1, 2)$. Hence A is non-identifiable. We exhibit the non-identifiability, as follows. Suppose s_1, s_2, s_3 follow exponential distributions with parameter 1, that r_1, r_2 follow standard Gaussian distributions, and that s_1, s_2, s_3, r_1, r_2 are independent. Then

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 - r_1 + r_2 \\ r_1 + r_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} s_1 + r_1 \\ s_2 \\ r_2 \end{pmatrix}.$$

Both source vectors have independent entries with the last coordinate Gaussian, since $r_1 + r_2$ and $-r_1 + r_2$ are independent Gaussians.

While Theorem 1.5 provides the if and only if theoretical criterion for when a matrix is identifiable, there is no known polynomial time algorithm for determining identifiability. Checking the criterion for a given $A \in \mathbb{R}^{I \times J}$ amounts to solving a system of degree two polynomials with $I - 1$ indeterminates, see Section 3. Solving this system symbolically via Gröbner basis has a complexity of $O(2^{2^{I-1}})$ [30], while solving numerically using monodromy solve and homotopy continuation has a lower bound of $\Omega(2^I)$ [15]. This highlights the importance of a practical rule of thumb that says whether to expect identifiability based on matrix size, when the matrix has no extra structure.

In this paper, a generic matrix is one that lies outside of a lower-dimensional set defined by the vanishing of certain equations. A property that holds for a generic matrix holds almost surely under any

continuous probability distribution on the space of matrices $\mathbb{R}^{I\times J}$ that assigns positive probability to every nonempty open set.

Theorem 1.9. *Let $A \in \mathbb{R}^{I\times J}$ be generic. Then*

1. *If $J \leq \binom{I}{2}$ or if $(I,J) = (2,2)$ or $(3,4)$, then A is identifiable;*
2. *If $J = \binom{I}{2} + 1$, where $I \geq 4$ and $I \equiv 2,3 \mod 4$, then there is positive probability that A is identifiable and a positive probability that A is non-identifiable;*
3. *If $J > \binom{I}{2} + 1$ or if $J = \binom{I}{2} + 1$ and $I \equiv 0,1 \mod 4$, then A is non-identifiable.*

To prove Theorem 1.9, we first studying the identifiability condition in Theorem 1.5 over the complex numbers. Then we specialize to the real numbers, proving the following. No extra complex solutions implies no extra real solutions. If there are a finite number k of extra complex solutions, the presence of extra real solutions depends on the parity of k: if odd, there is an extra solution, but if even, there may or may not be. If there are infinitely many extra complex solutions, then we show that there is an extra real solution.

Our third contribution is an algorithm to recover the mixing matrix $A \in \mathbb{R}^{I\times J}$ via coupled matrix and tensor decomposition, assuming the existence of the second and fourth order cumulant tensors. ICA has close connections to tensor decomposition. Comon explained how to obtain the mixing matrix from the higher order cumulant tensors of the observed variables [9]. When the sources are non-Gaussian and the number of sources is at most the number of observations, tensor algorithms include JADE [6], which uses fourth order cumulants, STOTD [14], which uses third order cumulants, and algorithms that combine cumulants of different orders [32]. For overcomplete ICA, algorithms include FOOBI [13], which uses fourth order cumulants and BIRTH, which uses hexacovariance (the flattening of the six order cumulant) [1].

In ICA, having a Gaussian source is natural to represent noise or, in practice, to allow sources that are close to Gaussian [42]. Gaussian distributions have all cumulants of order three or higher equal to zero. Thus, the column of the mixing matrix corresponding to the Gaussian source cannot be recovered via standard overcomplete ICA algorithms using tensor decomposition. Instead, recovering the mixing matrix is a coupled matrix and tensor decomposition problem. To the best of our knowledge, no existing algorithms address the case where one of the sources is Gaussian.

Our algorithm (see Algorithm 1) uses [26, Algorithm 1] in the first step. We could, in principle, use any tensor decomposition algorithm here. We use this method because of its compatibility with our second step: both look for rank one matrices or tensors in a linear space.

We apply our algorithm to synthetic data, where it corroborates our identifiability results, to the CIFAR-10 dataset [27] of images, to incorporate Gaussian noise into bases of image patches, cf. [35],

Algorithm 1 Recover A from the second and fourth order cumulants of $\mathbf{x}$

Input: Second and fourth order cumulants κ_2, κ_4 of $\mathbf{x} = A\mathbf{s}$ for matrix $A \in \mathbb{R}^{I\times J}$ and independent sources $\mathbf{s}$ with one Gaussian s_J.

1: **The first $J-1$ columns of A:** Compute the symmetric tensor decomposition of κ_4 to recover $\mathbf{a}_1, \ldots, \mathbf{a}_{J-1}$, up to permutation and scaling.

2: **The last column of A:** Find a rank one matrix in $\mathrm{Span}\{\mathbf{a}_1^{\otimes 2}, \ldots, \mathbf{a}_{J-1}^{\otimes 2}, \kappa_2\}$ that is not collinear to $\mathbf{a}_1^{\otimes 2}, \ldots, \mathbf{a}_{J-1}^{\otimes 2}$, by initializing at a random rank one matrix and a random point in the span and minimizing the distance between them using Powell's method [36].

Output: Matrix $A \in \mathbb{R}^{I\times J}$ with columns $\mathbf{a}_1, \ldots, \mathbf{a}_J$.

and to protein signaling [38], to incorporate Gaussian noise into the causal structure learning algorithm of [39].

The rest of the paper is organized as follows. We prove Theorem 1.5 in Section 2. We relate identifiability to systems of quadrics in Section 3 and study these quadrics in Section 4. We prove Theorem 1.9 in Section 5. Our numerical results are in Section 6. We also study the identifiability of special matrices in the Supplementary Material [45, Sections 1,2].

2. Characterization of identifiability

We prove Theorem 1.5, our characterization of the identifiability of ICA. One direction uses [16,25] to tackle the non-Gaussian sources and, for the Gaussian source, studies a system of degree two equations. The other direction is proved via a direct construction. The proof uses the second characteristic function, also known as cumulant generating function or log characteristic function.

Definition 2.1. The *second characteristic function* of a random variable $\mathbf{x}$, if existent, is

$$\Psi_{\mathbf{x}}(t) = \log \mathbb{E}[e^{i\mathbf{x}t}].$$

2.1. Sufficiency

We show that the condition of Theorem 1.5 is sufficient for identifiability of ICA.

Proposition 2.2. *Fix $A \in \mathbb{R}^{I\times J}$ with columns $\mathbf{a}_1, \ldots, \mathbf{a}_J$, and no pair of columns collinear. Then A is identifiable if the linear span of the rank one matrices $\mathbf{a}_1^{\otimes 2}, \ldots, \mathbf{a}_J^{\otimes 2}$ does not contain any real rank one matrix $\mathbf{b}^{\otimes 2}$, unless $\mathbf{b}$ is collinear to $\mathbf{a}_j$ for some $j \in \{1, \ldots, J\}$.*

For the proof, we use the second characteristic function, the cumulant generating function. We also use the following results: Theorem 2.3 relates the second characteristic function of a linear mixing to its sources, Theorem 2.4 settles the uniqueness of the mixing matrix of the non-Gaussian sources, and Theorem 2.5 turns the study of second characteristic functions into degree two equations.

Theorem 2.3 (See [10, Proposition 9.4]). *Let $\mathbf{x} = A\mathbf{s}$ with the entries of $\mathbf{s}$ independent. Then*

$$\Psi_{\mathbf{x}}(\mathbf{u}) = \sum_{j=1}^{J} \Psi_{s_j}(\mathbf{u}^\top \mathbf{a}_j),$$

in a neighborhood of the origin, where $\Psi_{\mathbf{x}}$ and Ψ_{s_j} are the second characteristic functions of the random variables $\mathbf{x}$ and s_j.

Theorem 2.4 (See [10, Theorem 9.1] and [25, Theorem 10.3.1]). *Let $\mathbf{x} = A\mathbf{s}$, where s_j are independent and non-degenerate and A does not have any collinear columns. Then $\mathbf{x} = A_1\mathbf{s_1} + A_2\mathbf{s_2}$, where $\mathbf{s_1}$ is a vector of non-Gaussian, $\mathbf{s_2}$ is a vector of Gaussian and independent of $\mathbf{s_1}$, and A_1 is unique, up to permuting and scaling of its columns.*

Theorem 2.5 (See [25, Lemma A.2.4]). *Fix vectors $\alpha_1, \dots, \alpha_n \in \mathbb{R}^p$ and a vector of variables $\mathbf{u} := (u_1, \dots, u_p)^\top$. Assume that α_i is not collinear to α_j for $i \neq j$ or to any elementary basis vector. Let $\psi_1, \dots, \psi_n, A_1, \dots, A_p$ be complex-valued continuous functions. Assume that*

$$\psi_1(\alpha_1^\top \mathbf{u}) + \cdots + \psi_n(\alpha_n^\top \mathbf{u}) = \sum_{i=1}^{p} A_i(u_i) + P_k(\mathbf{u}),$$

for all $\mathbf{u}$ in a neighborhood of the origin, where P_k is a degree k polynomial in $\mathbf{u}$. Then the functions ψ_j and A_i are all polynomials of degree at most $\max\{n, k\}$ in an interval around the origin.

Proof of Proposition 2.2. The model $\mathbf{x} = A\mathbf{s}$ is identifiable when $\mathbf{s}$ does not contain a Gaussian, by [16, Theorem 3], since A has no pair of columns collinear. It remains to consider the case that $\mathbf{s}$ contains one Gaussian. Without loss of generality, suppose that s_J is standard Gaussian. Take $B \in \mathbb{R}^{I \times K}$ with $K \leq J$ and $\mathbf{r} = (r_1, \dots, r_K)$ with $B\mathbf{r} = \mathbf{x}$. The columns of B corresponding to non-Gaussian sources must each be collinear to one of $\mathbf{a}_1, \dots, \mathbf{a}_{J-1}$, by Theorem 2.4. The vectors $\mathbf{r}$ and $\mathbf{s}$ have the same number of Gaussian entries. The number of non-Gaussian and Gaussian sources in $\mathbf{r}$ must then be $J-1$ and 1 respectively, by Theorem 2.4 and the assumption $K \leq J$. Without loss of generality, assume that r_J is a standard Gaussian. Then the first $J-1$ columns of B equal the first $J-1$ columns of A, up to scaling and permutation, by Theorem 2.4. Denote the last column of B by $\mathbf{b}$.

The second characteristic function of a standard Gaussian x is $\Psi_x(t) = -\frac{1}{2}t^2$. We have the equality

$$\sum_{j=1}^{J-1} (\Psi_{s_j} - \Psi_{r_j})(\mathbf{u}^\top \mathbf{a}_j) - \frac{1}{2}(\mathbf{u}^\top \mathbf{a}_J)^2 + \frac{1}{2}(\mathbf{u}^\top \mathbf{b})^2 = 0, \tag{4}$$

by Theorem 2.3. We apply Theorem 2.5 to the functions $\psi_j = \Psi_{s_j} - \Psi_{r_j}, A_i = 0, P_k = \frac{1}{2}(\mathbf{u}^\top \mathbf{a}_J)^2 - \frac{1}{2}(\mathbf{u}^\top \mathbf{b})^2$. It shows that the Taylor expansion of $\Psi_{s_j} - \Psi_{r_j}$ around the origin exists and is a polynomial, for all $1 \leq j \leq J-1$. Taking the degree two part of (4), we obtain an identity $\sum_{j=1}^{J-1} \lambda_j (\mathbf{u}^\top \mathbf{a}_j)^2 + (\mathbf{u}^\top \mathbf{a}_J)^2 - (\mathbf{u}^\top \mathbf{b})^2 = 0$, for some scalars $\lambda_1, \dots, \lambda_{J-1}$. That is,

$$\mathbf{b}^{\otimes 2} = \mathbf{a}_J^{\otimes 2} + \sum_{j=1}^{J-1} \lambda_j \mathbf{a}_j^{\otimes 2}.$$

By the condition in the statement, we conclude that $\mathbf{b}$ is collinear to $\mathbf{a}_j$ for some $j = 1, \dots, J$. However, $\mathbf{b}$ is not collinear to the first $J-1$ columns of B, since B has no pair of columns collinear. Hence, it is not collinear to the first $J-1$ columns of A, and therefore $\mathbf{b}$ is collinear to $\mathbf{a}_J$. Hence A and B are equal, up to permutation and scaling of columns. □

We now show how Proposition 2.2 suggests the viability of Algorithm 1.

Theorem 2.6. *Fix $J \leq \binom{I}{2}$. Suppose we have a generic $A \in \mathbb{R}^{I \times J}$ satisfying the condition in Theorem 1.5 and a system of independent sources $\mathbf{s}$ with one Gaussian and the rest non-Gaussian with non-vanishing second and fourth cumulants. Then A can be recovered, up to permutation and scaling of its columns, from the second and fourth cumulants of $A\mathbf{s}$.*

Proof. Without loss of generality, suppose that s_J is the Gaussian source. Let the fourth cumulant of s_j be $\lambda_j \in \mathbb{R}\backslash\{0\}$. Then the fourth cumulant tensor of $\mathbf{x} = A\mathbf{s}$ is $\kappa_4 = \sum_{j=1}^{J-1} \lambda_j \mathbf{a}_j^{\otimes 4}$. The rank is $J-1$, by

the genericity of the vectors $\mathbf{a}_j$. A generic symmetric tensor of format $I \times I \times I \times I$ has symmetric rank $\lceil \frac{1}{24}(I+3)(I+2)(I+1) \rceil$, by the Alexander-Hirschowitz Theorem [2]. The rank of κ_4 is less than the generic rank, since $J - 1 \leq \binom{I}{2}$ and the inequality $\binom{I}{2} < \lceil \frac{1}{24}(I+3)(I+2)(I+1) \rceil$ holds for all I. Hence κ_4 has a unique symmetric decomposition, by [8, Theorem 1.1]. Therefore, the first $J-1$ columns of A can be recovered, up to scaling and permutation, via the symmetric tensor decomposition of κ_4.

The second cumulant of $\mathbf{x}$ is $\kappa_2 = \sum_{j=1}^{J} \mu_k \mathbf{a}_j \otimes \mathbf{a}_j$, where μ_j is the variance of s_j. Hence $\mathrm{Span}\{\mathbf{a}_1^{\otimes 2}, \ldots, \mathbf{a}_{J-1}^{\otimes 2}, \kappa_2\} = \mathrm{Span}\{\mathbf{a}_1^{\otimes 2}, \ldots, \mathbf{a}_J^{\otimes 2}\}$. By assumption, the only real rank one matrices in $\mathrm{Span}\{\mathbf{a}_1^{\otimes 2}, \ldots, \mathbf{a}_J^{\otimes 2}\}$ are $\mathbf{a}_1^{\otimes 2}, \ldots, \mathbf{a}_J^{\otimes 2}$ up to scale, so a rank one matrix $\mathbf{a}_J^{\otimes 2}$ in $\mathrm{Span}\{\mathbf{a}_1^{\otimes 2}, \ldots, \mathbf{a}_{J-1}^{\otimes 2}, \kappa_2\}$ that is not collinear to $\mathbf{a}_1^{\otimes 2}, \ldots, \mathbf{a}_{J-1}^{\otimes 2}$ must be a scalar multiple of $\mathbf{a}_J^{\otimes 2}$. Hence, A is recovered uniquely up to column permutation and scaling from the second and fourth cumulant tensors of $A\mathbf{s}$. □

2.2. Necessity

We complete the proof of Theorem 1.5, showing that our condition is necessary.

Proposition 2.7. *Fix matrix $A \in \mathbb{R}^{I \times J}$ with columns $\mathbf{a}_1, \ldots, \mathbf{a}_J$. Then A is identifiable only if no pair of its columns is collinear and the linear span of $\mathbf{a}_1^{\otimes 2}, \ldots, \mathbf{a}_J^{\otimes 2}$ does not contain any real matrix $\mathbf{b}^{\otimes 2}$ unless $\mathbf{b}$ is collinear to $\mathbf{a}_j$ for some $j \in \{1, \ldots, J\}$.*

Proof. If the mixing matrix A has two collinear columns, we can combine them and obtain a matrix $B \in \mathbb{R}^{I \times (J-1)}$ and a system of independent sources $\mathbf{r} = (r_1, \ldots, r_{J-1})$ such that $B\mathbf{r}$ and $A\mathbf{x}$ have the same distribution. Hence identifiability implies no pair of collinear columns.

Assume there exists $\mathbf{b}^{\otimes 2} \in \mathrm{Span}\{\mathbf{a}_1^{\otimes 2}, \ldots, \mathbf{a}_J^{\otimes 2}\}$ with $\mathbf{b}$ not collinear to any $\mathbf{a}_j$. Then

$$\mathbf{b}^{\otimes 2} = \sum_{j=1}^{J-1} \lambda_j \mathbf{a}_j^{\otimes 2} + \mathbf{a}_J^{\otimes 2},$$

since we can assume without loss of generality that the coefficient of $\mathbf{a}_J$ is one. Let B be the matrix with columns $\mathbf{a}_1, \ldots, \mathbf{a}_{J-1}, \mathbf{b}$. We will construct independent random variables s_j and r_j such that $A\mathbf{s}$ and $B\mathbf{r}$ have the same distribution.

Let r_J and s_J be standard Gaussians. Choose $J-1$ non-Gaussian random variables $y_1, \ldots, y_{J-1}$ and Gaussian distributions $z_1, \ldots, z_{J-1}$ with second characteristic functions

$$\Psi_{z_j}(t) = \begin{cases} -\frac{1}{2}|\lambda_j| t^2 & \lambda_j \neq 0 \\ 0 & \lambda_j = 0, \end{cases}$$

such that these random variables together with the standard Gaussians r_J, s_J are independent. When $\lambda_j \geq 0$, set $r_j = y_j$ and $s_j = y_j + z_j$; when $\lambda_j < 0$, set $s_j = y_j$ and $r_j = y_j + z_j$. Then $s_1, \ldots, s_J$ are independent and $r_1, \ldots, r_J$ are independent. The source variables differ by Gaussians. We have

$$(\Psi_{s_j} - \Psi_{r_j})(t) = \begin{cases} (\Psi_{y_j + z_j} - \Psi_{y_j})(t) & \lambda_j > 0 \\ (\Psi_{y_j} - \Psi_{y_j + z_j})(t) & \lambda_j < 0 \\ 0 & \lambda_j = 0. \end{cases}$$

All three cases evaluate to give $(\Psi_{s_j} - \Psi_{r_j})(t) = -\frac{1}{2}\lambda_j t^2$. The second characteristic functions of $A\mathbf{s}, B\mathbf{r}$ are equal, by Theorem 2.3, since

$$\Psi_{A\mathbf{s}}(\mathbf{u}) - \Psi_{B\mathbf{r}}(\mathbf{u}) = \sum_{j=1}^{J-1} (\Psi_{s_j} - \Psi_{r_j})(\mathbf{u}^\mathsf{T}\mathbf{a}_j) - \frac{1}{2}(\mathbf{u}^\mathsf{T}\mathbf{a}_J)^2 + \frac{1}{2}(\mathbf{u}^\mathsf{T}\mathbf{b})^2 = 0. \tag{5}$$

Hence $A\mathbf{s}$ and $B\mathbf{r}$ have the same distribution. The last column of B is not collinear to any column of A, so A is not identifiable. □

Propositions 2.2 and 2.7 combine to prove Theorem 1.5. We give an example of an identifiable matrix A. We build more examples in the Supplementary Material [45].

Example 2.8. Let

$$A = \begin{pmatrix} 0 & 3 & 1 & \alpha & 1 & 0 & 0 & 0 \\ 1 & 9 & 11 & \beta & 0 & 1 & 0 & 0 \\ 2 & 14 & 13 & 17 & 0 & 0 & 1 & 0 \\ 3 & 1 & \gamma & 19 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\alpha = \frac{-27417}{160871}, \beta = \frac{282663}{36181}$ and $\gamma = \frac{-89735}{6339}$. If $\mathbf{b}^{\otimes 2} \in \mathrm{Span}\{\mathbf{a}_1^{\otimes 2}, \ldots, \mathbf{a}_8^{\otimes 2}\}$, then $\mathbf{b}$ collinear to $\mathbf{a}_j$ for some $j \in \{1, \ldots, 8\}$, as can be checked in Macaulay2 [18]. So A is identifiable, by Theorem 1.5. The matrix above has $I = 4$ and $J = 8$. It lies in the non-identifiable regime of Theorem 1.9, since a generic 4×8 mixing matrix is non-identifiable. However, the matrix A above is not generic.

3. From identifiability to systems of quadrics

Quadrics are homogeneous degree two polynomials. The characterization of identifiability in Theorem 1.5 is closely related to the study of systems of quadrics, as we now describe. We will use systems of quadrics to prove Theorem 1.9. The proof has two steps. The first step is to study the complex analogue of identifiability. The second step is to convert the complex results into real insights for the real setting of ICA. We give the complex analogue of Definition 1.3.

Definition 3.1. A mixing matrix $A \in \mathbb{C}^{I \times J}$ is *complex identifiable* if for any non-degenerate complex sources $\mathbf{s} = (s_1, \ldots, s_J)^\mathsf{T}$ with at most one s_j Gaussian, matrix A can be recovered uniquely, up to permutation and scaling of its columns, from $A\mathbf{s}$. That is, if $A\mathbf{s}$ and $B\mathbf{r}$ have the same distribution for some $B \in \mathbb{C}^{I \times K}$ with $K \leq J$ and some $\mathbf{r} = (r_1, \ldots, r_K)$, with the same number of Gaussian entries as $\mathbf{s}$, then $J = K$ and matrices A and B coincide, up to permutation and scaling of columns.

Complex ICA appears in applications to telecommunications [44] and in ICA algorithms such as [13] and [1]. We prove the complex analogue of Theorem 1.5.

Proposition 3.2. *A matrix $A \in \mathbb{C}^{I \times J}$ is* complex identifiable *if and only if no pair of its columns are collinear and the linear span of the matrices $\mathbf{a}_1^{\otimes 2}, \ldots, \mathbf{a}_J^{\otimes 2}$ does not contain any rank one matrix that is not collinear to $\mathbf{a}_1^{\otimes 2}, \ldots, \mathbf{a}_J^{\otimes 2}$.*

Proof. The sufficient direction is the same as the proof of Proposition 2.2. For the necessary direction, the proof is simpler than Proposition 2.7, since we allow complex square roots. The matrix A cannot

have collinear columns, as in Proposition 2.7. Given $\mathbf{b}^{\otimes 2} \in \text{Span}\{\mathbf{a}_j^{\otimes 2} : j = 1, \ldots, J\}$ such that $\mathbf{b}$ is not collinear to any $\mathbf{a}_j$, we write

$$\mathbf{b}^{\otimes 2} = \sum_{j=1}^{J-1} \lambda_j \mathbf{a}_j^{\otimes 2} + \mathbf{a}_J^{\otimes 2},$$

since we can assume without loss of generality that the coefficient of $\mathbf{a}_J$ is one. Define A and B as in the proof of Proposition 2.7. Let r_J and s_J be standard Gaussians. Choose $J-1$ non-Gaussian random variables $y_1, \ldots, y_{J-1}$ and standard Gaussian distributions $z_1, \ldots, z_{J-1}$. Define $r_j = y_j$ and $s_j = y_j + \sqrt{\lambda_j} z_j$. Then $\Psi_{s_j}(t) = \Psi_{y_j}(t) + \lambda_j \Psi_{z_j}(t)$. The source variables differ by a complex scalar multiple of a Gaussian. The second characteristic functions for $A\mathbf{s}$ and $B\mathbf{r}$ are equal, by Theorem 2.3, since

$$\Psi_{A\mathbf{s}}(\mathbf{u}) - \Psi_{B\mathbf{r}}(\mathbf{u}) = \sum_{j=1}^{J-1} (\Psi_{s_j} - \Psi_{r_j})(\sum_{i=1}^{I} u_i a_{ij}) - \frac{1}{2}(\sum_{i=1}^{I} u_i a_{iJ})^2 + \frac{1}{2}(\sum_{i=1}^{I} u_i b_i)^2 = 0. \tag{6}$$

Hence $A\mathbf{s}$ and $B\mathbf{r}$ have the same distribution. The last column of B is not collinear to any column of A, so A is not complex identifiable. □

We will prove the following characterization of complex identifiability.

Theorem 3.3. *Let $A \in \mathbb{C}^{I \times J}$ be generic. Then*

1. *If $J \leq \binom{I}{2}$ or if $(I, J) = (2, 2)$ or $(3, 4)$, then A is complex identifiable;*
2. *If $J \geq \binom{I}{2} + 2$ or if $J \geq \binom{I}{2} + 1$ for $I \geq 4$, then A is complex non-identifiable.*

Theorem 3.3 immediately implies part 1 of Theorem 1.9.

Corollary 3.4. *Let $A \in \mathbb{R}^{I \times J}$ be generic. If $J \leq \binom{I}{2}$ or if $(I, J) = (2, 2)$ or $(3, 4)$, then A is identifiable.*

Proof. Such matrices A are complex identifiable, by Theorem 3.3. Hence no pair of its columns are collinear and the linear span of $\mathbf{a}_1^{\otimes 2}, \ldots, \mathbf{a}_J^{\otimes 2}$ does not contain any rank one matrix not collinear to $\mathbf{a}_1^{\otimes 2}, \ldots, \mathbf{a}_J^{\otimes 2}$, by Proposition 3.2. In particular, the linear span contains no real rank one matrix. Hence A is identifiable, by Theorem 1.5. □

Theorem 1.5 and Proposition 3.2 translate to conditions on systems of quadrics, as we now explain. Theorem 1.5 involves the linear space $\text{Span}\{\mathbf{a}_j^{\otimes 2} : j = 1, \ldots, J\}$. We view a symmetric matrix M either as an array of $I \times I$ entries M_{ij}, for $1 \leq i, j \leq I$, or as a vector of $\binom{I+1}{2}$ entries M_{ij}, for $1 \leq i \leq j \leq I$. In our identifiability conditions, two vectors or matrices are equivalent if they agree up to scale, so it is convenient to work in projective space.

Definition 3.5. The projective space $\mathbb{P}(V)$ is the set of equivalence classes of nonzero vectors in the linear space V, where two vectors are identified if one is a nonzero scalar multiple of the other. The space $\mathbb{P}(V)$ is called the projectivization of V and it has dimension $\dim V - 1$. When $V \cong \mathbb{C}^m$, we often denote $\mathbb{P}(V)$ by $\mathbb{P}_{\mathbb{C}}^{m-1}$ or simply $\mathbb{P}^{m-1}$.

We denote the projectivization of $\mathrm{Span}\{\mathbf{a}_j^{\otimes 2} : j = 1, \ldots, J\}$ by $\mathcal{W}(A)$. It is a linear space in $\mathbb{P}_{\mathbb{C}}^{m-1}$, where $m = \binom{I+1}{2}$. The coordinates on $\mathbb{P}_{\mathbb{C}}^{m-1}$ are $\mathbf{z} = (z_{ij} : 1 \le i \le j \le I)$. The space $\mathcal{W}(A)$ is defined by linear relations

$$l_1(\mathbf{z}) = \sum_{1 \le i \le j \le I} \lambda_{ij}^{(1)} z_{ij} \quad \ldots \quad l_k(\mathbf{z}) = \sum_{1 \le i \le j \le I} \lambda_{ij}^{(k)} z_{ij}. \tag{7}$$

The number of linear relations k is the number of linearly independent conditions that cut out $\mathcal{W}(A)$. In particular, if $\mathcal{W}(A)$ spans the whole space then $k = 0$. We study rank one matrices in $\mathcal{W}(A)$.

An algebraic variety is the set of solutions to a system of polynomial equations in an algebraically closed field such as $\mathbb{C}$. The set of rank one $I \times I$ symmetric matrices is a variety.

Definition 3.6 (See [19, Examples 2.4 and 18.13]). Let $m = \binom{I+1}{2}$. The *second Veronese variety* for $\mathbb{C}^I$ is the image of the map

$$\phi : \mathbb{C}^I \to \mathbb{C}^m$$
$$(x_1, \ldots, x_I) \mapsto (x_1^2 : x_1 x_2 : \ldots : x_I^2).$$

We index the coordinates of $\mathbb{C}^m$ by $(z_{11}, z_{12}, \ldots, z_{II})$. The second Veronese variety is cut out by the 2×2 minors of the corresponding rank one matrix: $\{z_{ij} z_{kl} - z_{ik} z_{jl} : 1 \le i, j, k, l \le I\}$. We denote the projectivization of the second Veronese variety by $\mathcal{V}_I$. It is the set of equivalence classes of rank one $I \times I$ symmetric matrices up to nonzero scalar multiple. We also call it the second Veronese embedding or the second Veronese variety of $\mathbb{P}_{\mathbb{C}}^{I-1}$.

The intersection $\mathcal{V}_I \cap \mathcal{W}(A)$ consists of all rank one matrices, up to scale, that lie in $\mathrm{Span}\{\mathbf{a}_j^{\otimes 2} : j = 1, \ldots, J\}$. In particular, it contains $\mathbf{a}_1^{\otimes 2}, \ldots, \mathbf{a}_J^{\otimes 2}$. The rank one condition converts (7) into the system of quadrics

$$f_1(\mathbf{x}) = \sum_{1 \le i \le j \le I} \lambda_{ij}^{(1)} x_i x_j \quad \ldots \quad f_k(\mathbf{x}) = \sum_{1 \le i \le j \le I} \lambda_{ij}^{(k)} x_i x_j. \tag{8}$$

The intersection $\mathcal{V}_I \cap \mathcal{W}(A)$ is the vanishing locus of the quadrics $f_1, \ldots, f_k$, which we denote by $V(f_1, \ldots, f_k)$. We say $\{f_1, \ldots, f_k\}$ is a system of quadrics defining $\mathcal{V}_I \cap \mathcal{W}(A)$. Proposition 3.2 says that A is complex identifiable if and only if $V(f_1, \ldots, f_k) = \{\mathbf{a}_1^{\otimes 2}, \ldots, \mathbf{a}_J^{\otimes 2}\}$. Theorem 1.5 says that A is identifiable if and only if $V(f_1, \ldots, f_k)$ does not contain any *real* points other than $\mathbf{a}_1^{\otimes 2}, \ldots, \mathbf{a}_J^{\otimes 2}$.

Example 3.7. Let A be the matrix from Example 1.7. The linear equations defining $\mathcal{W}(A)$ are the rows of the matrix

$$\begin{array}{c} \begin{matrix} z_{11} & z_{12} & z_{13} & z_{14} & z_{22} & z_{23} & z_{24} & z_{33} & z_{34} & z_{44} \end{matrix} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \end{array}$$

The corresponding system of quadrics defining $\mathcal{V}_I \cap \mathcal{W}(A)$ is obtained by replacing z_{ij} by $x_i x_j$, to give $f_1 = x_1 x_2 - x_2 x_3 + x_2 x_4$, $f_2 = x_1 x_3 - x_2 x_3 + x_2 x_4$, $f_3 = x_1 x_2 - x_2 x_3 + x_3 x_4$, $f_4 = x_1 x_4$.

4. Systems of quadrics

Quadrics have been studied as far back as 300BC [21]. They remain a popular topic in algebraic geometry, see e.g. [5,17,33]. In this section, we prove results for systems of quadrics, which may be of independent interest, and which are building blocks of our proof of Theorem 1.9. We prove the following quadric restatement of Theorem 3.3 in Section 4.1.

Theorem 4.1. *Let $\mathcal{V}_I$ be the second Veronese embedding of $\mathbb{P}^{I-1}_{\mathbb{C}}$. Suppose that $\mathbf{v}_1^{\otimes 2}, \dots, \mathbf{v}_J^{\otimes 2}$ are generic points on $\mathcal{V}_I$ with $\mathcal{W}(A)$ their projective linear span. Let the system of quadrics defining $\mathcal{V}_I \cap \mathcal{W}(A)$ be $\{f_1, \dots, f_k\}$, with vanishing locus $V(f_1, \dots, f_k)$.*

1. *If $J \leq \binom{I}{2}$ or if $(I, J) = (2, 2)$ or $(3, 4)$, then $V(f_1, \dots, f_k) = \{\mathbf{v}_1^{\otimes 2}, \dots, \mathbf{v}_J^{\otimes 2}\}$.*
2. *If $J > \binom{I}{2} + 1$ or if $J > \binom{I}{2}$ for $I \geq 4$, then $V(f_1, \dots, f_k) \supsetneq \{\mathbf{v}_1^{\otimes 2}, \dots, \mathbf{v}_J^{\otimes 2}\}$.*

Example 4.2. We revisit Examples 1.7 and 3.7. A Macaulay2 [18] computation confirms that $V(f_1, f_2, f_3, f_4) = \{\mathbf{a}_1^{\otimes 2}, \dots, \mathbf{a}_6^{\otimes 2}\}$, which proves that A is complex identifiable (and, in particular, identifiable), by Theorem 4.1.

We prove the following result in Section 4.2. It is used in the identifiability result for $J = \binom{I}{2} + 1$ in Theorem 1.9. By an open set of systems of $I - 1$ quadrics, we mean the coefficients of the quadrics form an open set in the space of $(I - 1)\binom{I+1}{2}$ coefficients.

Theorem 4.3. *For every even integer ℓ from 0 to 2^{I-1}, there is an open set of systems of $I - 1$ quadrics in $\mathbb{R}[x_1, \dots, x_I]$ that have 2^{I-1} distinct intersection points, of which ℓ are real.*

4.1. Complex solutions to a system of quadrics

In this section, we prove Theorems 4.1 and 3.3. As above, let $\mathcal{V}_I$ denote the second Veronese embedding of $\mathbb{P}^{I-1}_{\mathbb{C}}$ in $\mathbb{P}^{m-1}_{\mathbb{C}}$, where $m = \binom{I+1}{2}$. We use the notions of dimension and degree for algebraic varieties, see [19, Lectures 11 and 18] for definitions.

Lemma 4.4. *Let $\mathbf{v}_1^{\otimes 2}, \dots, \mathbf{v}_J^{\otimes 2}$ be generic points of $\mathcal{V}_I$. Then the matrix in $\mathbb{C}^{\binom{I+1}{2} \times J}$ with columns $\mathbf{v}_1^{\otimes 2}, \dots, \mathbf{v}_J^{\otimes 2}$ has full rank.*

Proof. The highest rank is attained for generic matrices, so it suffices to exhibit an example with full rank. Suppose $\mathbf{e}_1, \dots, \mathbf{e}_I$ are canonical basis vectors in $\mathbb{C}^I$. Let $S = \{\mathbf{e}_1^{\otimes 2}, \dots, \mathbf{e}_I^{\otimes 2}\} \cup \{(\mathbf{e}_i + \mathbf{e}_j)^{\otimes 2} : i < j\}$. Then $|S| = m = \binom{I+1}{2}$. A subset of S of size $J \leq m$ is linearly independent. When $J > m$, taking the union of S with any $J - m$ symmetric rank one matrices forms a linear space of dimension m. In both cases, the matrix has full rank. □

Lemma 4.5. *For $\binom{I}{2} + 1$ generic points in $\mathcal{V}_I$, with span $\mathcal{W}$, the intersection $\mathcal{V}_I \cap \mathcal{W}$ consists of 2^{I-1} distinct points.*

Proof. The variety $\mathcal{V}_I$ has dimension $I - 1$ and degree 2^{I-1} [19, Exercise 2.8 and Example 18.13]. A generic linear space of codimension $I - 1$ therefore intersects $\mathcal{V}_I$ in 2^{I-1} distinct points, by Bézout's Theorem. Our goal is to show that $\mathcal{W}$ from the statement is sufficiently generic: a codimension $I - 1$ subspace that intersects $\mathcal{V}_I$ in degree many distinct points.

Let $J = \binom{I}{2} + 1$ and $m = \binom{I+1}{2}$. The space $\mathcal{W}$ is spanned by J points, and lives in $\mathbb{P}^{m-1}_{\mathbb{C}}$. It has projective dimension $(J-1)$, by Lemma 4.4. Hence it has codimension $I-1$, since $(I-1)+(J-1) = m-1$. That is, for generic $\mathbf{v}_1, \ldots, \mathbf{v}_J \in \mathbb{P}^{I-1}_{\mathbb{C}}$, the projective linear space $\mathrm{Span}\{\mathbf{v}_1^{\otimes 2}, \ldots, \mathbf{v}_J^{\otimes 2}\}$ is an element of the Grassmannian variety of $(J-1)$-dimensional projective linear spaces in $\mathbb{P}^{m-1}_{\mathbb{C}}$, which we denote $\mathrm{Gr}(J-1, m-1)$.

Let $\mathcal{U} \subset \mathrm{Gr}(J-1, m-1)$ be the set of spaces spanned by J points on $\mathcal{V}_I$. The set $\mathcal{U}$ is open and dense in $\mathrm{Gr}(J-1, m-1)$, as follows. For a generic $(J-1)$-dimensional linear space $\mathcal{L}$, the intersection $\mathcal{V}_I \cap \mathcal{L}$ spans $\mathcal{L}$, by [19, Proposition 18.10] applied $I-1$ times, since the variety $\mathcal{V}_I$ is irreducible and non-degenerate (not contained in any hyperplane) and its generic hyperplane sections are also non-degenerate and irreducible if $\dim \mathcal{V}_I \geq 2$. Choosing a basis of $\mathcal{L}$ from $\mathcal{V}_I \cap \mathcal{L}$ shows that $\mathcal{U}$ is open and dense.

Let $\mathcal{U}'$ denote the elements of $\mathrm{Gr}(J-1, m-1)$ that intersect $\mathcal{V}_I$ in degree many distinct points. Then $\mathcal{U}'$ is open and dense, by Bézout's theorem. Hence $\mathcal{U} \cap \mathcal{U}'$ is open and dense in $\mathrm{Gr}(J-1, m-1)$. Define the map

$$\Phi : (\mathbb{P}^{I-1}_{\mathbb{C}})^J \dashrightarrow \mathrm{Gr}(J-1, m-1)$$
$$(\mathbf{v}_1, \ldots, \mathbf{v}_J) \mapsto \mathrm{Span}\{\mathbf{v}_1^{\otimes 2}, \ldots, \mathbf{v}_J^{\otimes 2}\},$$

where Span here denotes the projective span. It is defined almost everywhere, by Lemma 4.4. The pre-image $\Phi^{-1}(\mathcal{U} \cap \mathcal{U}')$ consists of collections of points for which $\mathcal{V}_I \cap \mathcal{W}$ is 2^{I-1} distinct points. As the pre-image of a dense open set, it is dense and open in $(\mathbb{P}^{I-1}_{\mathbb{C}})^J$. □

We use the following algebraic geometry result to prove Theorem 4.1.

Theorem 4.6 (Generalized Trisecant Lemma, see [7, Proposition 2.6]). *Let $X \subseteq \mathbb{P}^{m-1}_{\mathbb{C}}$ be an irreducible, reduced, non-degenerate projective variety of dimension $I-1$ and let J be a non-negative integer with $(J-1)+(I-1) < m-1$. Let $P_1, \ldots, P_J$ be general points on X. Then the intersection of X with the subspace spanned by $P_1, \ldots, P_J$ is the points $P_1, \ldots, P_J$.*

Proof of Theorem 4.1. Let $m = \binom{I+1}{2}$. Assume that $J > \binom{I}{2} + 1$. For generic $\mathbf{v}_1, \ldots, \mathbf{v}_J \in \mathbb{P}^{I-1}_{\mathbb{C}}$, the space $\mathcal{W} = \mathrm{Span}\{\mathbf{v}_1^{\otimes 2}, \ldots, \mathbf{v}_J^{\otimes 2}\}$ has projective codimension at most $I-2$, by Lemma 4.4. The dimension of $V(f_1, \ldots, f_k)$ is therefore at least $(I-1)-(I-2) = 1$, by Krull's Principal Ideal Theorem [20, Theorem 1.11A]. Hence there are infinitely many points in $\mathcal{V}_I \cap \mathcal{W}$, so $V(f_1, \ldots, f_k) \supsetneq \{\mathbf{v}_1^{\otimes 2}, \ldots, \mathbf{v}_J^{\otimes 2}\}$.

Assume $J = \binom{I}{2} + 1$. For generic $\mathbf{v}_1^{\otimes 2}, \ldots, \mathbf{v}_J^{\otimes 2}$, the intersection $\mathcal{V}_I \cap \mathcal{W}$ consists of 2^{I-1} distinct points, by Lemma 4.5. When $I \geq 4$, we have $2^{I-1} > J$, hence $V(f_1, \ldots, f_k) \supsetneq \{\mathbf{v}_1^{\otimes 2}, \ldots, \mathbf{v}_J^{\otimes 2}\}$. When $I \leq 3$, we have $2^{I-1} = J$, so $V(f_1, \ldots, f_k) = \{\mathbf{v}_1^{\otimes 2}, \ldots, \mathbf{v}_J^{\otimes 2}\}$.

It remains to consider $J \leq \binom{I}{2}$. The Veronese variety $\mathcal{V}_I \in \mathbb{P}^{m-1}$ is irreducible, reduced and non-degenerate with dimension $I-1$. We have $(J-1)+(I-1) < m-1$. Hence $V(f_1, \ldots, f_k) = \{\mathbf{v}_1^{\otimes 2}, \ldots, \mathbf{v}_J^{\otimes 2}\}$ for generic $\mathbf{v}_1, \ldots, \mathbf{v}_J \in \mathbb{P}^{I-1}_{\mathbb{C}}$, by Theorem 4.6. See also [26, Proposition 3.2]. □

Proof of Theorem 3.3. Let $\mathbf{v}_1, \ldots, \mathbf{v}_J$ in Theorem 4.1 be $\mathbf{a}_1, \ldots, \mathbf{a}_J$. Theorem 3.3 is equivalent to the statement about quadrics in Theorem 4.1, see the end of Section 3. □

4.2. Real solutions to a system of quadrics

In this section, we prove Theorem 4.3.

Proof of Theorem 4.3. A system of $I-1$ homogeneous quadrics in $\mathbb{R}[x_1,\dots,x_I]$ generically has 2^{I-1} complex solutions. There is a dense open set of quadric systems whose solution set consists of 2^{I-1} distinct complex points. The number of real solutions is constant on the connected components of this set. Hence it suffices to find one system with ℓ distinct real solutions for each even $0 \le \ell \le 2^{I-1}$. There is a dense open set of quadric systems such that any solution has $x_I \neq 0$. Without loss of generality, we dehomogenize the quadrics, intersecting them with the plane $x_I = 1$. Then it suffices to find $I-1$ inhomogeneous quadrics in $\mathbb{R}[x_1,\dots,x_{I-1}]$ that intersect in 2^{I-1} distinct points with ℓ distinct real solutions for each even $0 \le \ell \le 2^{I-1}$. We prove this by induction.

When $I = 2$, we have a single univariate quadric $g(x) = ax^2 + bx + c$. It generically has two distinct roots; there are 0 or 2 real roots, depending on the sign of $b^2 - 4ac$.

Assume the result for I: there is a system of $I-2$ quadrics in $\mathbb{R}[x_1,\dots,x_{I-2}]$ with 2^{I-2} distinct solutions and ℓ of them real, for all even values $0 \le \ell \le 2^{I-2}$. Choose a real value α such that no solution has $x_1 = \alpha$. Then adding the quadric $(x_1-\alpha)^2 - x_{I-1}^2$ gives $I-1$ quadrics in $\mathbb{R}[x_1,\dots,x_{I-1}]$ with 2^{I-1} distinct solutions, of which 2ℓ are real. It remains to find a system of quadrics with 2^{I-1} distinct solutions, of which $2\ell-2$ are real, for every even ℓ in the range $2 \le \ell \le 2^{I-2}$. Consider our system of $I-2$ quadrics with 2^{I-2} distinct solutions, of which ℓ are real. We can apply a change of basis to ensure that the x_1 coordinates of the roots have distinct values, since the roots are distinct. Choose $\beta \in \mathbb{R}$ in between the largest and second largest x_1 values that appear among the ℓ real roots. Then add the quadric $-\beta^2 + x_1^2 + x_{I-1}^2$. The resulting system has all solutions distinct and $2\ell-2$ of them real. □

5. From complex to real identifiability

We specialize from complex to real identifiability to prove Theorem 1.9. Results to study the real solutions are in Section 5.1 and the proof of Theorem 1.9 is in Section 5.2.

5.1. The projected second Veronese

We introduce the projected second Veronese variety and compute its dimension and degree.

Changing basis on $\mathbb{R}^I$ does not affect the identifiability of $A \in \mathbb{R}^{I\times J}$. That is, when $J \ge I$, if A is identifiable, so is MA for all invertible $M \in \mathbb{R}^{I\times I}$. We can therefore assume without loss of generality that a generic $A \in \mathbb{R}^{I\times J}$ has the form

$$A = \begin{pmatrix} \vdots & & \vdots & \vdots & & \vdots \\ \mathbf{a}_1 & \cdots & \mathbf{a}_{J-I} & \mathbf{e}_1 & \cdots & \mathbf{e}_I \\ \vdots & & \vdots & \vdots & & \vdots \end{pmatrix}. \tag{9}$$

This motivates the following definition.

Definition 5.1 (The projected second Veronese variety). Consider the map $\varphi : \mathbb{C}^I \to \mathbb{C}^{\binom{I}{2}}$ with $\varphi(x_1,\dots,x_I) = (x_1x_2,\dots,x_{I-1}x_I)$ and the projection map $\pi : (\mathbb{C}^\star)^{\binom{I}{2}} \to \mathbb{P}_{\mathbb{C}}^{\binom{I}{2}-1}$. The I-th projected second Veronese embedding, denoted $\mathcal{Z}_I$, is the closure of $\pi \circ \varphi((\mathbb{C}^\star)^I)$ in $\mathbb{P}_{\mathbb{C}}^{\binom{I}{2}-1}$.

Proposition 5.2. *Let $A \in \mathbb{R}^{I\times J}$ have the form (9). Let $\mathcal{W}(A)_\pi$ be the projective linear space spanned by*

$$\varphi(\mathbf{a}_1), \ldots, \varphi(\mathbf{a}_{J-I}).$$

Then A is identifiable if and only if no pair of its columns are collinear and the only real points in the intersection $\mathcal{W}(A)_\pi \cap \mathcal{Z}_I$ are $\varphi(\mathbf{a}_1), \ldots, \varphi(\mathbf{a}_{J-I})$.

Proof. The matrix A is identifiable if and only if no pair of its columns are collinear and the real points in the intersection $\mathcal{W}(A) \cap \mathcal{V}_I$ are $\mathbf{a}_1^{\otimes 2}, \ldots, \mathbf{a}_{J-I}^{\otimes 2}, \mathbf{e}_1^{\otimes 2}, \ldots, \mathbf{e}_I^{\otimes 2}$, by Theorem 1.5. The span of $\mathbf{e}_1^{\otimes 2}, \ldots, \mathbf{e}_I^{\otimes 2}$ is the diagonal matrices. Hence, $\mathbf{b}^{\otimes 2}$ lies in $\mathcal{W}(A) \cap \mathcal{V}_I$ if and only if its off-diagonal part $\varphi(\mathbf{b})$ lies in $\mathcal{W}(A)_\pi \cap \mathcal{Z}_I$. □

Lemma 5.3. *The projected second Veronese variety $\mathcal{Z}_I$ has dimension $I-1$ and degree $2^{I-1}-I$.*

Proof. We use the Hilbert polynomial to compute the dimension and degree of $\mathcal{Z}_I$, see [20, Section 1.7]. Let $h(\ell)$ be the dimension of degree ℓ polynomials in the coordinate ring $\mathbb{C}[\mathcal{Z}_I]$. These are degree 2ℓ polynomials obtained from products of $x_1x_2, \ldots, x_{I-1}x_I$. Thus, if $x_i^{\ell+1}$ divides a monomial, it cannot be degree 2ℓ in $\mathbb{C}[\mathcal{Z}_I]$. A monomial $x_1^{a_1}x_2^{a_2}\cdots x_I^{a_I}$ is in $\mathbb{C}[\mathcal{Z}_I]$ if and only if $a_1 + \ldots + a_I = 2\ell$ with $1 \le a_i \le \ell$ for all i. Hence

$$h(\ell) = \binom{2\ell + I - 1}{I-1} - I\binom{I+\ell-1}{I-1},$$

a polynomial in ℓ with leading term $\frac{2^{I-1}-I}{(I-1)!}\ell^{I-1}$. Hence $\dim \mathcal{Z}_I = I-1$ and $\deg \mathcal{Z}_I = 2^{I-1}-I$. □

5.2. Generic identifiability

In this section, we prove Theorem 1.9.

Lemma 5.4. *If a generic matrix in $\mathbb{R}^{I\times J}$ is non-identifiable, then a generic matrix in $\mathbb{R}^{I\times J'}$ is non-identifiable for all $J' > J$.*

Proof. Fix a generic matrix $A \in \mathbb{R}^{I\times J'}$. The submatrix consisting of the first J columns of A is a generic $I \times J$ matrix, hence is non-identifiable by assumption. So, the intersection $\mathrm{Span}\{\mathbf{a}_j^{\otimes 2} : 1 \le j \le J\} \cap \mathcal{V}_I$ contains a real point that is not collinear to any of $\{\mathbf{a}_j^{\otimes 2} : 1 \le j \le J\}$, by Theorem 1.5. This point is not collinear to any column of A, by genericity. □

Proposition 5.5. *For $I \equiv 0, 1 \mod 4$ and $J = \binom{I}{2} + 1$, a generic $A \in \mathbb{R}^{I\times J}$ is non-identifiable.*

Proof. The intersection $\mathcal{V}_I \cap \mathcal{W}(A)$ consists of 2^{I-1} distinct points, by Lemma 4.5. Complex intersection points come in pairs, since $\mathcal{V}_I \cap \mathcal{W}(A)$ is the vanishing locus of quadrics with real coefficients. So there is an even number of real points in $\mathcal{V}_I \cap \mathcal{W}(A)$. There are J real points, which correspond to the columns of A. If $I \equiv 0, 1 \mod 4$, then $J = \binom{I}{2} + 1$ is odd. Hence there is an extra real solution, so A is not identifiable, by Theorem 1.5. □

Combining the above with Lemma 5.4 gives the following.

Corollary 5.6. *When $I \equiv 0, 1 \mod 4$ and $J \ge \binom{I}{2} + 1$, a generic $A \in \mathbb{R}^{I\times J}$ is non-identifiable.*

Recall the map from Definition 5.1, with $\varphi(x_1, \ldots, x_I) = (x_1x_2, \ldots, x_{I-1}x_I)$. We study generic $A \in \mathbb{R}^{I\times J}$ via the images under φ of $J - I$ generic vectors, by Proposition 5.2.

Lemma 5.7. *Let $\mathcal{I}$ be a homogeneous ideal generated by polynomials with real coefficients and X the vanishing locus of $\mathcal{I}$. If X has odd degree, then it contains a real point. If moreover* $\dim X \geq 1$*, then X contains infinitely many real points.*

Proof. The ideal $\mathcal{I}$ is generated by polynomials with real coefficients, so complex solutions come in pairs. If $\dim X = 0$, then X contains a real point. If $d = \dim X \geq 1$, a generic real linear space of codimension d intersects X to give an odd number of points, so X contains a real point. Assume for contradiction that X contains only finitely many real points. There is a generic real codimension d linear space that does not pass through these points, but that intersects X in degree many points. This intersection contributes a new real point, a contradiction. □

Proposition 5.8. *If $I \equiv 2, 3 \mod 4$ and $J > \binom{I}{2} + 1$, a generic $A \in \mathbb{R}^{I\times J}$ is non-identifiable.*

Proof. Let $X := \mathcal{W}(A)_\pi \cap \mathcal{Z}_I$. The matrix of first $J - I$ columns of A is generic, so $\mathcal{W}(A)_\pi$ has projective dimension $J - I - 1$. Hence $\dim X = (I - 1) + (J - I - 1) - (\binom{I}{2} - 1) > 0$ and $\deg X = 2^{I-1} - I$, by similar arguments as Lemma 4.5.

When $I \equiv 3 \mod 4$, the degree of X is odd. Hence X contains infinitely many real points, by Lemma 5.7. It remains to consider $I \equiv 2 \mod 4$. If $J = \binom{I}{2} + 2$, we consider the system of quadrics in $I - 1$ variables obtained by setting $x_I = 0$. Denote the quadrics by $g_1, \ldots, g_\ell$, where ℓ is the codimension of $\mathcal{W}(A)_\pi$. Its vanishing locus consists of $2^{I-2} - (I - 1)$ points, by Lemma 5.3 and similar arguments to Lemma 4.5. Since $2^{I-2} - (I - 1)$ is odd, there is a real point $\varphi(y_1, \ldots, y_{I-1})$ in the intersection, by Lemma 5.7. Hence $\varphi(y_1, \ldots, y_{I-1}, 0) \in \mathcal{W}(A)_\pi \cap \mathcal{Z}_I$. This point is not collinear to any column of A, by genericity. The case $J > \binom{I}{2} + 2$ follows from Lemma 5.4. □

The cases remaining are $I \equiv 2, 3 \mod 4$, $I \geq 4$ and $J = \binom{I}{2} + 1$. The following result follows from Theorem 3.3. We will use it to prove these remaining cases.

Corollary 5.9. *Let $m = \binom{I+1}{2}$. For a generic linear space $\mathcal{W} \subseteq \mathbb{P}^{m-1}_{\mathbb{C}}$ of dimension $\binom{I}{2}$, any $\binom{I}{2} + 1$ points in the intersection $\mathcal{W} \cap \mathcal{V}_I$ are linearly independent as affine vectors.*

Proof. Fix a generic linear space $\mathcal{W}$ of dimension $\binom{I}{2}$. It intersects $\mathcal{V}_I$ in 2^{I-1} distinct points. The intersection points span $\mathcal{W}$, by [19, Proposition 18.10] applied $I - 1$ times.

Assume for contradiction that there is a set S of $\binom{I}{2} + 1$ points in $\mathcal{W} \cap \mathcal{V}_I$ that are linearly dependent. They span a linear space $\mathcal{W}'$ of dimension $\ell < \binom{I}{2} + 1$. Choose a subset of S of size ℓ that spans $\mathcal{W}'$. These points define a matrix $A \in \mathbb{C}^{I\times\ell}$ that is not complex identifiable, by Proposition 3.2.

Let S_ℓ be the sets of ℓ linearly independent vectors in $\mathbb{P}^{I-1}_{\mathbb{C}}$ whose corresponding matrices in $\mathbb{C}^{I\times\ell}$ are complex non-identifiable. Complex identifiability holds generically, by Theorem 3.3, since $\ell < \binom{I}{2} + 1$. Hence $\dim S_\ell < \dim((\mathbb{P}^{I-1}_{\mathbb{C}})^\ell) = \ell(I - 1)$.

Define the continuous map ϕ_ℓ that sends a collection of $\binom{I}{2} + 1$ vectors, the first ℓ of which are in S_ℓ, to the linear space their second outer products span. Then $\mathcal{W}$ is in the image of ϕ_ℓ, since it is spanned by the ℓ points spanning $\mathcal{W}'$ plus $\binom{I}{2} + 1 - \ell$ other points. The dimension of $\operatorname{im}\phi_\ell$ is at most $\dim S_\ell + (I - 1)(\binom{I}{2} + 1 - \ell) < (I - 1)(\binom{I}{2} + 1)$. But the space of $\binom{I}{2}$ dimensional spaces in $\mathbb{P}^{m-1}_{\mathbb{C}}$ has dimension $(I - 1)(\binom{I}{2} + 1)$, by [19, Lecture 6], a contradiction. □

Corollary 5.9 is still true for a generic real linear space, since a generic real linear space of dimension n is a generic complex linear space of dimension n.

Proposition 5.10. *Let $I \equiv 2, 3 \mod 4$, $I \geq 4$ and $J = \binom{I}{2} + 1$. For matrices in $\mathbb{R}^{I \times J}$, identifiability and non-identifiability both occur with positive probability.*

Proof of of Proposition 5.10. We construct non-empty open sets of identifiable and non-identifiable matrices. More specifically, we find open sets U_1 and U_2 in $\mathbb{R}^{I \times J}$ such that for each matrix in U_1, the corresponding system of quadrics has J real solutions, and the system of quadrics for U_2 has $J+2$ real solutions. To find U_1 and U_2, we construct a continuous map from matrices to quadric systems and use Theorem 4.3.

We construct a map that sends a matrix $A \in \mathbb{R}^{I \times J}$ to the system of quadrics that define $\mathcal{W}(A) \cap \mathcal{V}_I$. The map is continuous on the dense set of full rank matrices A with $\dim \mathcal{W}(A) = J - 1$ and such that $\mathcal{W}(A)$ intersects $\mathcal{V}_I$ generically transversely. There is a continuous function that sends a projective $(J-1)$-dimensional linear space to a choice of $I-1$ linear relations defining it, e.g. using the orthogonal complement. We compose it with the map that sends the linear relation $\sum \lambda_{ij} z_{ij}$ to the quadric $\sum \lambda_{ij} x_i x_j$. Finally, we pre-compose it with the continuous map that sends $A \in \mathbb{R}^{I \times J}$ to $\mathcal{W}(A)$. Call the resulting map ψ.

There exist open sets $\mathcal{U}_1$ (respectively $\mathcal{U}_2$) in $(\mathbb{P}^{m-1}_{\mathbb{R}})^{I-1}$ such that the $I-1$ quadrics intersect in 2^{I-1} distinct points with J (respectively $J+2$) of them real, by Theorem 4.3. Among these real solutions, J will generically be linearly independent, by Corollary 5.9. Hence there exists $A \in \mathbb{R}^{I \times J}$ in the preimage of ψ. Set $U_i = \psi^{-1}(\mathcal{U}_i)$ for $i = 1, 2$. □

Proof of Theorem 1.9. Corollary 3.4 gives the first part. Proposition 5.10 gives the second part. The third part follows from Corollary 5.6 and Proposition 5.8. □

6. Numerical experiments

We evaluate the performance of Algorithm 1 on synthetic and real data. The code for our computations can be found in the supplementary material [45].

The second and fourth cumulant tensors κ_2, κ_4 are the input to Algorithm 1. For synthetic data, these are either true population cumulants or sample cumulants. For real data, the tensors are obtained from samples. The first step of Algorithm 1 computes the symmetric tensor decomposition of the fourth cumulant κ_4, using [26, Algorithm 1]. Only the columns corresponding to the non-Gaussian sources (i.e. the first $J-1$ columns) can be recovered since the Gaussian source has zero fourth cumulant. The outputs are unit vectors $\mathbf{a}_1, \ldots, \mathbf{a}_{J-1}$. For the second step of Algorithm 1, we decompose the matrix κ_2 using $\mathbf{a}_1^{\otimes 2}, \ldots, \mathbf{a}_{J-1}^{\otimes 2}$ and an unknown symmetric rank-1 matrix. We minimize

$$\min_{\mathbf{v} \in \mathbb{R}^I, l \in \mathbb{R}^J} \|\kappa_2 - \sum_{j=1}^{J-1} l_j \mathbf{a}_j^{\otimes 2} - l_J \mathbf{v}^{\otimes 2}\|,$$

using Powell's method [36]. We initialize at a random unit vector $\mathbf{v} \in \mathbb{R}^I$ and a random vector $l \in \mathbb{R}^J$. We normalize the output $\mathbf{v}$ and set it to be the last column $\mathbf{a}_J$.

We usually use $1000(I+J)$ iterations for the minimization with Powell's method, the default in the python function `scipy.optimize.minimize`. For synthetic datasets on small sample size, and for real data, we increase the number of iterations and run the minimization 10 times and select the best solution. That is, from 10 outputs $(\mathbf{v}_1, l_1), \ldots, (\mathbf{v}_{10}, l_{10})$, we choose the one with the smallest value of $\|\kappa_2 - \sum_{j=1}^{J-1} (l_i)_j \mathbf{a}_j^{\otimes 2} - (l_i)_J \mathbf{v}_i^{\otimes 2}\|$.

6.1. Synthetic data

We take as input a matrix $A \in \mathbb{R}^{I \times J}$ with its columns rescaled to unit vectors, for various I and J. Assume that the first $J-1$ columns correspond to the non-Gaussian sources, and that the last column corresponds to the Gaussian source. We compute the cumulants in one of two ways:

1. Use the population cumulants, $\kappa_2 = \sum_{j=1}^{J} \sigma_j \mathbf{a}_j^{\otimes 2}$ and $\kappa_4 = \sum_{j=1}^{J-1} \lambda_j \mathbf{a}_j^{\otimes 4}$, where σ_j is the variance of source j and λ_j is its fourth cumulant.
2. Fix sources $\mathbf{s}$ and compute cumulants from samples of $A\mathbf{s}$.

The output of Algorithm 1 is a matrix $A' \in \mathbb{R}^{I \times J}$ with unit vector columns. The last column corresponds to the Gaussian source.

We measure the proximity of A and A'. Since identifiability is only up to permutation and rescaling, we allow for re-ordering of the first $J-1$ columns. Rather than searching over all ways to match the first $J-1$ columns of A to those of A', we use a greedy algorithm to approximate the matching, as follows. We fix the first column of A, denoted $\mathbf{a}_1$. We choose one of the first $J-1$ columns of A' whose cosine similarity with $\mathbf{a}_1$ has largest absolute value. We set this to be the first column of A' (changing its sign if the cosine similarity is negative). Then we select among the remaining $J-2$ columns, the one with the largest absolute cosine similarity with $\mathbf{a}_2$ and set this as the second column of A' (again, changing the sign if the cosine similarity is negative). We continue until we reach the last column. Then we compute the relative Frobenius error

$$\sqrt{\sum_{i=1}^{I}\sum_{j=1}^{J}(a_{ij} - a'_{ij})^2}/J.$$

We study a range of I and J, using the population cumulants in Figure 2. We examine how the error changes with the variance of the Gaussian source in Figure 3. We test how our algorithm performs with sample cumulant tensors in Figure 4.

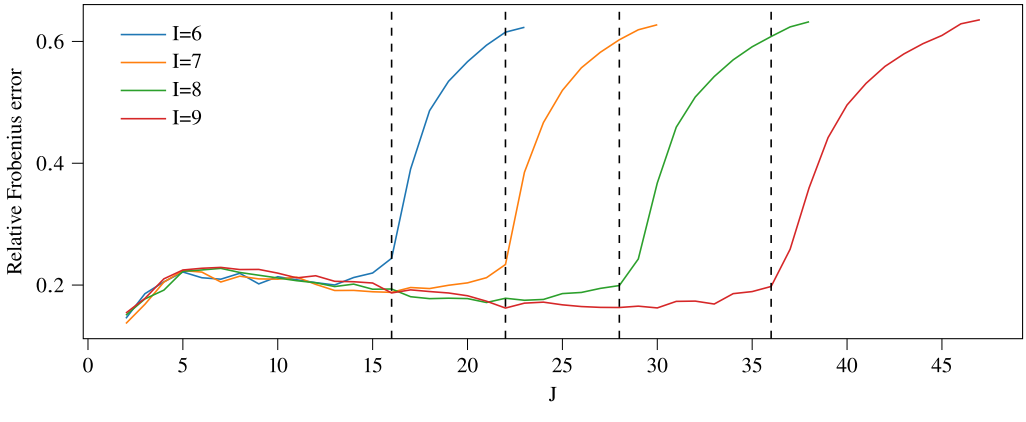

Figure 2. Relative Frobenius error using population cumulants. We fix the fourth cumulant of the non-Gaussian sources to be 6, the second cumulant to be 1, and consider a standard Gaussian as the Gaussian source. We run 1000 experiments on each pair (I, J) and plot the mean relative Frobenius error. The black dashed lines are the identifiability thresholds from Theorem 1.9: $\binom{I}{2} + 1$ for $I = 6, 7$ and $\binom{I}{2}$ for $I = 8, 9$. The errors are low for J below the threshold and increase beyond it. The small increase in error from $J = \binom{I}{2}$ to $\binom{I}{2} + 1$ for $I = 6, 7$ is due to the positive probability of non-identifiability when $J = \binom{I}{2} + 1$, see Theorem 1.9.

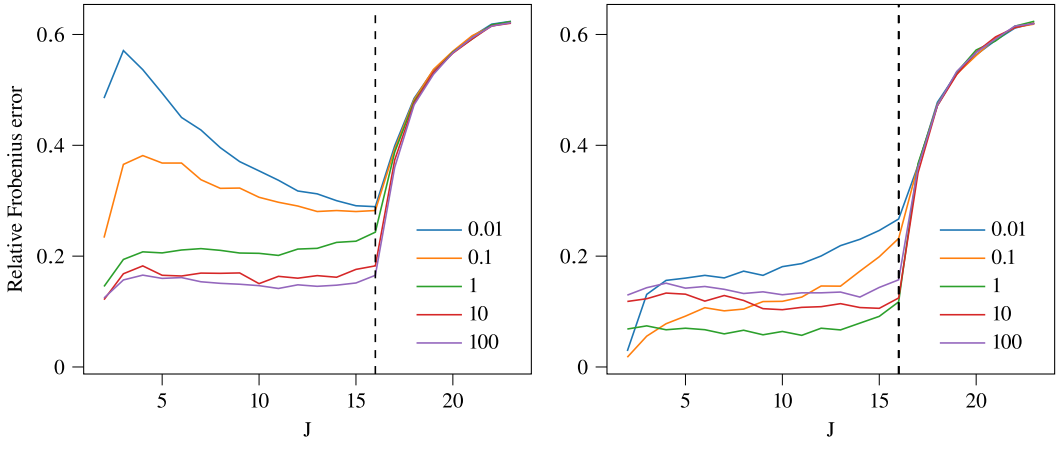

Figure 3. Relative Frobenius error for differing Gaussian source variance. We consider variances in the range $\{0.01, 0.1, 1, 10, 100\}$. We fix $I = 6$. The black dashed lines are the threshold $J = \binom{I}{2} + 1 = 16$. For each matrix size and variance, we run the experiment 1000 times and plot the mean. As the variance of the Gaussian source increases, the relative Frobenius error decreases. This is expected, as a higher variance makes the Gaussian component more easily distinguished from the other signals, which leads to improved accuracy in the second step of Algorithm 1. In the left figure, we use $1000(I + J)$ iterations in Powell's method. On the right, we increase the number of iterations to 500000, which makes the algorithm more stable to change of variance.

Our experiments indicate that the robustness of the algorithm depends on two main factors, the variance of the Gaussian source and the sample size: larger Gaussian variance improves identifiability by making the Gaussian component more easily distinguished from other sources (see Figure 3) and

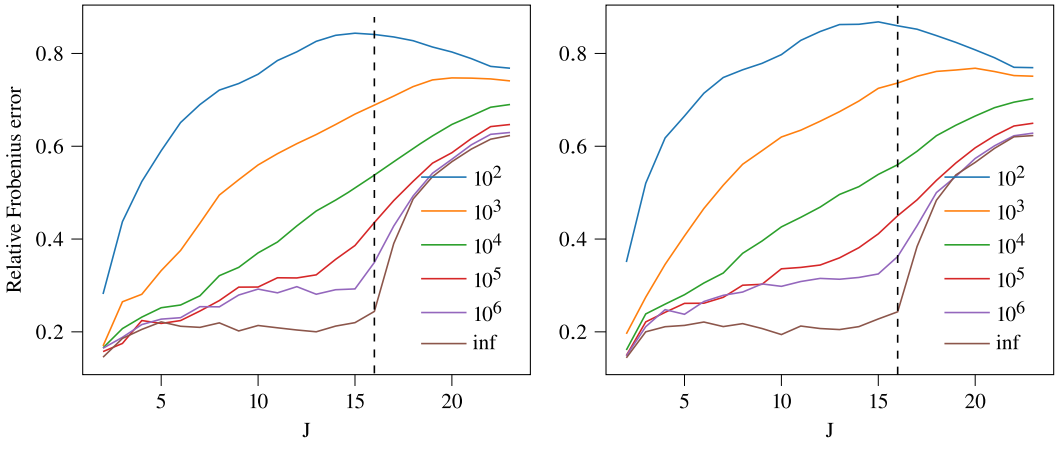

Figure 4. Relative Frobenius error with sample cumulant tensors. We take our non-Gaussian sources to be exponential sources with parameter 1 (left) and Student t-distributed sources with five degrees of freedom (right). We set the Gaussian source to be a standard Gaussian. We fix $I = 6$. For each pair (I, J), we run 1000 experiments and plot the mean Frobenius error. In both plots, the error decreases as the sample size increases. We plot the population cumulant method (labelled as 'inf') for comparison.

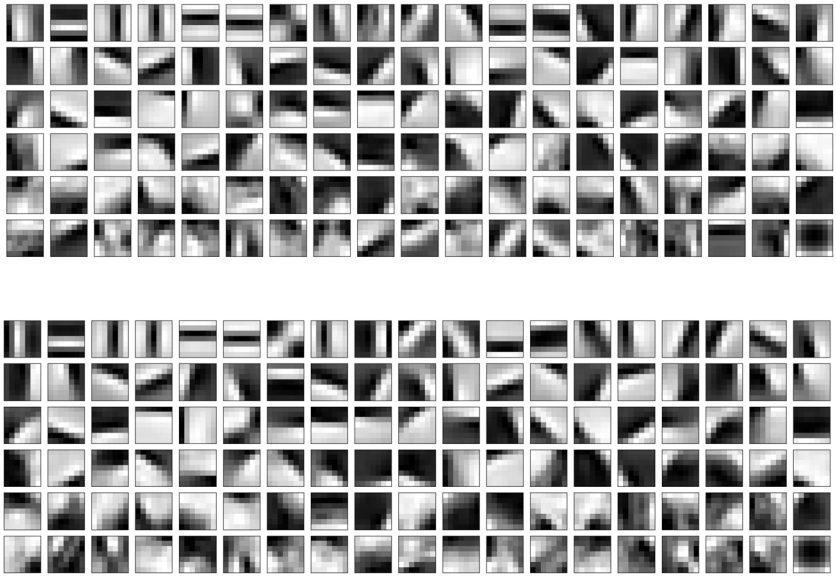

Figure 5. We divide the 7×7 images in half to give two datasets, each with $400000 = \frac{1}{2}(16 \times 50000)$ datapoints of dimension 49, keeping the number of images from each class roughly the same between the two halves. We apply Algorithm 1 to the two datasets and assess the similarity of the output. We obtain two matrices $A \in \mathbb{R}^{49 \times J}$, where J is the number of sources. We illustrate the results for $J = 114$. The columns of the two 49×114 matrices are plotted as grayscale 7×7 images. We observe the visual agreement of the 114 images, reflecting the identifiability. The last image is the Gaussian source, the Gaussian noise in the images. The two Gaussian sources have cosine similarity 0.99. Their grayscale plots show that pixels patterns have more Gaussian noise at the center than the edges.

error decreases as sample size increases (see Figure 4). We leave as directions for future work the theoretical understanding of the sample complexity of the algorithm its error analysis.

6.2. Image data

We test our algorithm on the CIFAR-10 dataset [27], following [35]. We define a training set of 50000 color images, each of size 32×32, in one of 10 classes. We convert each image to grayscale and divided the central 28×28 image into 16 images of size 7×7. Our assumption is that there is a collection of 7×7 images, from which the others are expressible as a linear combination. We use Algorithm 1 to plot the columns of the $49 \times J$ mixing matrix, see Figure 5.

6.3. Protein data

We fit an adapted LiNGAM model [39] to a single-cell flow cytometry dataset [38]. Each datapoint measures 11 proteins in a cell. Suppose that the 11 proteins are $X_1, \ldots, X_{11}$ and that G is a directed acyclic graph with nodes $X_1, \ldots, X_{11}$ whose edges $E(G)$ indicates causal relationships with weights λ_{ij}

on the edge $j \to i$. A linear structural equation model writes

$$X_i = \sum_{(j \to i) \in E(G)} \lambda_{ij} X_j + e_i.$$

The LiNGAM algorithm learns the graph G from the higher-order cumulants of X, assuming the noise terms e_i are non-Gaussian, using ICA. We use our algorithm for ICA with a Gaussian source to adapt the LiNGAM to allow a latent source of Gaussian noise:

$$X_i = \sum_{(j \to i) \in E(G)} \lambda_{ij} X_j + e_i + t_i y,$$

where y is a Gaussian variable and t_i is its effect on variable X_i. Let $\Lambda \in \mathbb{R}^{11 \times 11}$ be the matrix of weights, with (i, j) entry λ_{ij}. Then

$$\mathbf{X} = \Lambda \mathbf{X} + \mathbf{e} + \mathbf{t}y \quad \Longrightarrow \quad \mathbf{X} = (I - \Lambda)^{-1}\mathbf{e} + (I - \Lambda)^{-1}\mathbf{t}y = ((I - \Lambda)^{-1} | (I - \Lambda)^{-1}\mathbf{t}) \begin{pmatrix} \mathbf{e} \\ y \end{pmatrix}.$$

Algorithm 1 recovers $A \in \mathbb{R}^{11 \times 12}$ with first 11 columns $(I - \Lambda)^{-1}$ and last column $(I - \Lambda)^{-1}\mathbf{t}$. As in the LiNGAM algorithm, this enables us to recover the directed acyclic graph G. But we also recover the vector $\mathbf{t}$, which measures the Gaussian noise effect for each X_i.

There are samples collected under 13 different perturbations in [38]. To test our adapted LiNGAM model via Algorithm 1, we divide the data from each perturbation into half to form two datasets. We log transform the data. We run our algorithm on each half of the data and compute the similarity of the Gaussian source effects, using cosine similarity, see Figure 6. We plot the Gaussian source effects in Figure 7.

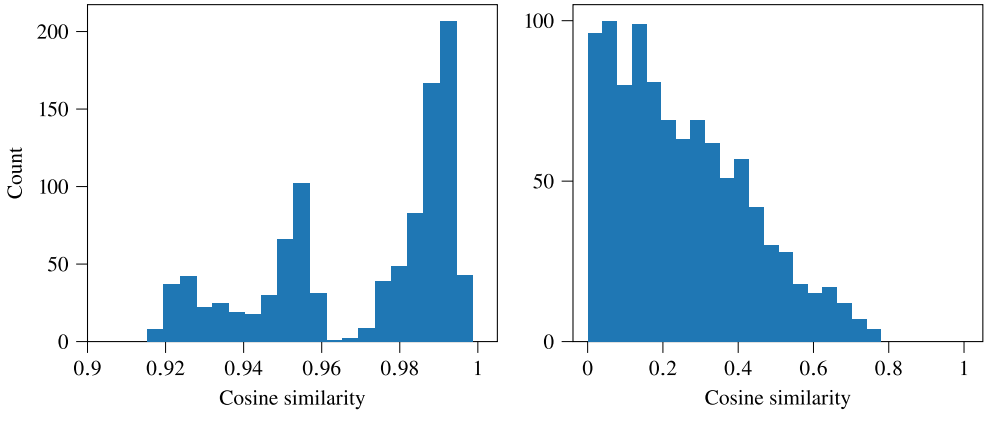

Figure 6. For 1000 experiments, we plot a histogram of the cosine similarity of the Gaussian column of the mixing matrices (left). For comparison, we show the plot for two random matrices (right). Randomness comes from both tensor decomposition and the optimization step. The plot validates the choice of our adapted LiNGAM model, since the Gaussian source effects are consistent between experiments.

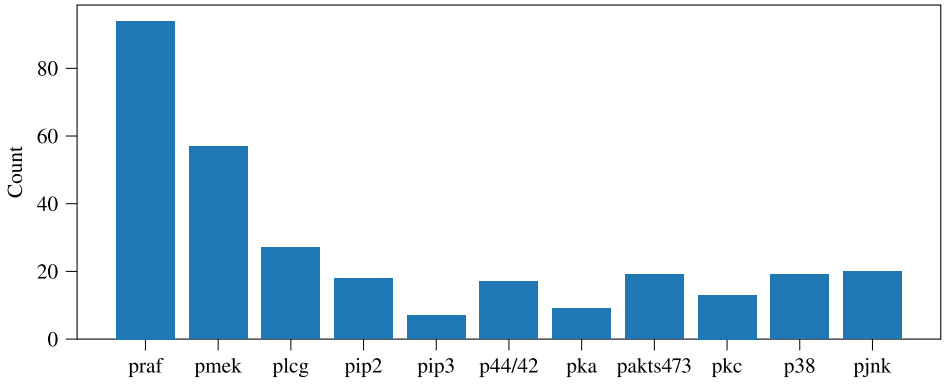

Figure 7. We fit our adapted LiNGAM model and focus on how the Gaussian source effects each protein. The vector $\mathbf{t}$ quantifies the effect of the latent Gaussian noise y on each of the 11 proteins. Because the model output has randomness, we repeat the procedure 100 times. In each run, we rank the entries of $\mathbf{t}$ by absolute value. We record the number of times each protein appears among the top three. The results show that the log-transformed measurements of the proteins praf and pmek most frequently appear among the top three entries of $\mathbf{t}$, indicating that they are influenced more by the shared Gaussian noise than the other proteins.

7. Conclusion

In this paper, we characterized the identifiability of overcomplete ICA. For generic mixing, we saw how identifiability is determined by the number of sources and the number of observations. We gave a coupled matrix and tensor decomposition algorithm for recovering the mixing matrix from the second and fourth cumulants and tested it on real and simulated data. Our algorithm allows for a Gaussian source, which is not true of other algorithms for ICA or overcomplete ICA.

We conclude by mentioning directions for future study.

- The work [3] studies the sample complexity of cumulant tensors and the estimation rate of the mixing matrix in the undercomplete setting. Build on the present results to study these in the overcomplete setting.
- Theorem 1.9 gives three possibilities for generic identifiability: the middle case has a positive probability of identifiability and of non-identifiability. Compute these probabilities for a suitable distribution of mixing matrices.
- Adapt Theorem 1.9 and Algorithm 1 to incorporate structure on A, such as sparsity.
- Extend the complex identifiability results such as [45, Theorem 1.1] to the real setting.
- Study the special loci of identifiable and non-identifiable matrices geometrically, e.g. compute the dimension and degree of their Zariski closures.

Sections 4 and 5 studied the real algebraic geometry question of whether a linear space spanned by real points on the second Veronese variety only intersects the variety in exactly these real points. In the subsequent work [37], we generalized this from the second Veronese variety to other real varieties and obtained a similar trichotomy as in Theorem 1.9.

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Supplementary Material

Supplement to "Identifiability of overcomplete independent component analysis" (DOI: 10.3150/25-BEJ1929SUPP; .pdf). In the supplement [45], we study special matrices: non-identifiable matrices in the range of (I, J) where identifiability generically holds, and identifiable matrices in the range of (I, J) where non-identifiability generically holds. We also provide the code, graphs, and data for computational results of the paper.

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